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# Links between Discriminating and Identifying Codes in the Binary Hamming Space

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## Abstract

Let  $F^n$  be the binary  $n$ -cube, or binary Hamming space of dimension  $n$ , endowed with the Hamming distance, and  $\mathcal{E}^n$  (respectively,  $\mathcal{O}^n$ ) the set of vectors with even (respectively, odd) weight. For  $r \geq 1$  and  $x \in F^n$ , we denote by  $B_r(x)$  the ball of radius  $r$  and centre  $x$ . A code  $C \subseteq F^n$  is said to be  $r$ -identifying if the sets  $B_r(x) \cap C$ ,  $x \in F^n$ , are all nonempty and distinct. A code  $C \subseteq \mathcal{E}^n$  is said to be  $r$ -discriminating if the sets  $B_r(x) \cap C$ ,  $x \in \mathcal{O}^n$ , are all nonempty and distinct. We show that the two definitions, which were given for general graphs, are equivalent in the case of the Hamming space, in the following sense: for any odd  $r$ , there is a bijection between the set of  $r$ -identifying codes in  $F^n$  and the set of  $r$ -discriminating codes in  $F^{n+1}$ .

**Key Words:** Graph Theory, Coding Theory, Discriminating Codes, Identifying Codes, Hamming Space, Hypercube

## 1 Introduction

We define identifying and discriminating codes in a connected, undirected graph  $G = (V, E)$ , in which a *code* is simply a nonempty subset of vertices. These definitions can help, in various meanings, to unambiguously determine a vertex. The motivations may come from processor networks where we wish to locate a faulty vertex under certain conditions, or from the need to identify an individual, given its set of attributes.

In  $G$  we define the usual distance  $d(v_1, v_2)$  between two vertices  $v_1, v_2 \in V$  as the smallest possible number of edges in any path between them. For an integer  $r \geq 0$  and a vertex  $v \in V$ , we define  $B_r(v)$  the *ball* of radius  $r$  centred at  $v$ , as the set of vertices within distance  $r$  from  $v$ . Whenever two vertices  $v_1$  and  $v_2$  are such that  $v_1 \in B_r(v_2)$  (or, equivalently,  $v_2 \in B_r(v_1)$ ), we say that they *r-cover* each other. A set  $X \subseteq V$  *r-covers* a set  $Y \subseteq V$  if every vertex in  $Y$  is *r-covered* by at least one vertex in  $X$ .

The elements of a code  $C \subseteq V$  are called *codewords*. For each vertex  $v \in V$ , we denote by

$$K_{C,r}(v) = C \cap B_r(v)$$

the set of codewords *r-covering*  $v$ . Two vertices  $v_1$  and  $v_2$  with  $K_{C,r}(v_1) \neq K_{C,r}(v_2)$  are said to be *r-separated* by code  $C$ , and any codeword belonging to exactly one of the two sets  $B_r(v_1)$  and  $B_r(v_2)$  is said to *r-separate*  $v_1$  and  $v_2$ .

A code  $C \subseteq V$  is called *r-identifying* [10] if all the sets  $K_{C,r}(v)$ ,  $v \in V$ , are nonempty and distinct. In other words, every vertex is *r-covered* by at least one codeword, and every pair of vertices is *r-separated* by at least one codeword. Such codes are also sometimes called *differentiating dominating sets* [8].

We now suppose that  $G$  is bipartite:  $G = (V = I \cup A, E)$ , with no edges inside  $I$  nor  $A$  — here,  $A$  stands for *attributes* and  $I$  for *individuals*. A code  $C \subseteq A$  is said to be *r-discriminating* [4] if all the sets  $K_{C,r}(i)$ ,  $i \in I$ , are nonempty and distinct. From the definition we see that we can consider only odd values of  $r$ .

In the following, we drop the general case and turn to the binary Hamming space of dimension  $n$ , also called the binary  $n$ -cube, which is a regular bipartite graph. First we need to give some specific definitions and notation.

We consider the  $n$ -cube as the set of binary row-vectors of length  $n$ , and as so, we denote it by  $G = (F^n, E)$  with  $F = \{0, 1\}$  and  $E = \{\{x, y\} : d(x, y) = 1\}$ , the usual graph distance  $d(x, y)$  between two vectors  $x$  and  $y$

being called here the *Hamming distance* — it simply consists of the number of coordinates where  $x$  and  $y$  differ. The *Hamming weight* of a vector  $x$  is its distance to the all-zero vector, i.e., the number of its nonzero coordinates. A vector is said to be *even* (respectively, *odd*) if its weight is even (respectively, odd), and we denote by  $\mathcal{E}^n$  (respectively,  $\mathcal{O}^n$ ) the set of the  $2^{n-1}$  even (respectively, odd) vectors in  $F^n$ . Without loss of generality, for the definition of an  $r$ -discriminating code, we choose the set  $A$  to be  $\mathcal{E}^n$ , and the set  $I$  to be  $\mathcal{O}^n$ . Additions are carried coordinatewise and modulo two.

Given a vector  $x \in F^n$ , we denote by  $\pi(x)$  its parity-check bit:  $\pi(x) = 0$  if  $x$  is even,  $\pi(x) = 1$  if  $x$  is odd. Therefore, if  $|$  stands for concatenation of vectors,  $x|\pi(x)$  is an even vector. Finally, we denote by  $M_r(n)$  (respectively,  $D_r(n)$ ) the smallest possible cardinality of an  $r$ -identifying (respectively,  $r$ -discriminating) code in  $F^n$ .

In Section 2, we show that in the particular case of Hamming space, the two notions of  $r$ -identifying and  $r$ -discriminating codes actually coincide for all odd values of  $r$  and all  $n \geq 2$ , in the sense that there is a bijection between the set of  $r$ -identifying codes in  $F^n$  and the set of  $r$ -discriminating codes in  $F^{n+1}$ .

## 2 Identifying is discriminating

As we now show with the following two theorems, for any odd  $r \geq 1$ , any  $r$ -identifying code in  $F^n$  can be extended into an  $r$ -discriminating code in  $F^{n+1}$ , and any  $r$ -discriminating code in  $F^n$  can be shortened into an  $r$ -identifying code in  $F^{n-1}$ . First, observe that  $r$ -identifying codes exist in  $F^n$  if and only if  $r < n$ .

**Theorem 1** *Let  $n \geq 2, p \geq 0$  be such that  $2p + 1 < n$ , let  $C \subseteq F^n$  be a  $(2p + 1)$ -identifying code and let*

$$C' = \{c|\pi(c) : c \in C\}.$$

*Then  $C'$  is  $(2p + 1)$ -discriminating in  $F^{n+1}$ . Therefore,*

$$D_{2p+1}(n+1) \leq M_{2p+1}(n). \tag{1}$$

**Proof.** Let  $r = 2p + 1$ . By construction,  $C'$  contains only even vectors. We shall prove that (a) any odd vector  $x \in \mathcal{O}^{n+1}$  is  $r$ -covered by at least one codeword of  $C'$ ; (b) given any two distinct odd vectors  $x, y \in \mathcal{O}^{n+1}$ , there is at least one codeword in  $C'$  which  $r$ -separates them.

(a) We write  $x = x_1|x_2$  with  $x_1 \in F^n$  and  $x_2 \in F$ . Because  $C$  is  $r$ -identifying in  $F^n$ , there is a codeword  $c \in C$  with  $d(x_1, c) \leq r$ . Let  $c' = c|\pi(c)$ .

If  $d(x_1, c) \leq r - 1$ , then whatever the values of  $x_2$  and  $\pi(c)$  are, we have  $d(x, c') \leq r$ ; we assume therefore that  $d(x_1, c) = r = 2p + 1$ , which

implies that  $x_1$  and  $c$  have different parities. Since  $x_1|x_2$  and  $c|\pi(c)$  also have different parities, we have  $x_2 = \pi(c)$  and  $d(x, c') = r$ . So the codeword  $c' \in C'$   $r$ -covers  $x$ .

(b) We write  $x = x_1|x_2$ ,  $y = y_1|y_2$ , with  $x_1, y_1 \in F^n$ ,  $x_2, y_2 \in F$ . Since  $C$  is  $r$ -identifying in  $F^n$ , there is a codeword  $c \in C$  which is, say, within distance  $r$  from  $x_1$  and not from  $y_1$ :  $d(x_1, c) \leq r$ ,  $d(y_1, c) > r$ . Let  $c' = c|\pi(c)$ .

For the same reasons as above,  $x$  is within distance  $r$  from  $c'$ , whereas obviously,  $d(y, c') \geq d(y_1, c) > r$ . So  $c' \in C'$   $r$ -separates  $x$  and  $y$ .

Inequality (1) follows.  $\square$

**Theorem 2** *Let  $n \geq 3, p \geq 0$  be such that  $2p + 2 < n$ , let  $C \subseteq \mathcal{E}^n$  be a  $(2p + 1)$ -discriminating code and let  $C' \subseteq F^{n-1}$  be any code obtained by the deletion of one coordinate in  $C$ . Then  $C'$  is  $(2p + 1)$ -identifying in  $F^{n-1}$ . Therefore,*

$$M_{2p+1}(n-1) \leq D_{2p+1}(n). \quad (2)$$

**Proof.** Let  $r = 2p + 1$ . Let  $C \subseteq \mathcal{E}^n$  be an  $r$ -discriminating code and  $C' \subseteq F^{n-1}$  be the code obtained by deleting, say, the last coordinate in  $C$ . We shall prove that (a) any vector  $x \in F^{n-1}$  is  $r$ -covered by at least one codeword of  $C'$ ; (b) given any two distinct vectors  $x, y \in F^{n-1}$ , there is at least one codeword in  $C'$  which  $r$ -separates them.

(a) The vector  $x|(\pi(x) + 1) \in F^n$  is odd. As such, it is  $r$ -covered by a codeword  $c = c'|u \in C \subseteq \mathcal{E}^n$ :  $c' \in C'$ ,  $u = \pi(c')$ , and  $d(x|(\pi(x) + 1), c) \leq r$ . This proves that  $x$  is within distance  $r$  from a codeword of  $C'$ .

(b) Both  $x|(\pi(x) + 1)$  and  $y|(\pi(y) + 1)$  are odd vectors in  $F^n$ , and there is a codeword  $c = c'|u \in C \subseteq \mathcal{E}^n$ , with  $c' \in C'$ ,  $u = \pi(c')$ , which  $r$ -separates them: without loss of generality,  $d(x|(\pi(x) + 1), c) \leq r$  whereas  $d(y|(\pi(y) + 1), c)$ , which is an odd integer, is at least  $r + 2$ . Then obviously,  $d(x, c') \leq r$  and  $d(y, c') \geq r + 1$ , i.e., there is a codeword in  $C'$  which  $r$ -separates  $x$  and  $y$ .

Inequality (2) follows.  $\square$

**Corollary 3** *For all  $n \geq 2$  and  $p \geq 0$  such that  $2p + 1 < n$ , we have:*

$$D_{2p+1}(n+1) = M_{2p+1}(n).$$

$\square$

### 3 Conclusion

We have shown the equivalence between discriminating and identifying codes; the latter being already well studied, this entails a few consequences on discriminating codes.

For example, the complexity of problems on discriminating codes is the same as that for identifying codes; in particular, it is known [9] that deciding whether a given code  $C \subseteq F^n$  is  $r$ -identifying is co-NP-complete.

For yet another issue, constructions, we refer to, e.g., [1]–[3], [6], [9], [10] or [11]; visit also [12]. In the recent [7], tables for exact values or bounds on  $M_1(n)$ ,  $2 \leq n \leq 19$ , and  $M_2(n)$ ,  $3 \leq n \leq 21$ , are given.

Discriminating codes have not been thoroughly studied so far; let us simply mention [4] for a general introduction and [5] in the case of planar graphs.

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