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Abstract

We study the asymptotic behavior of wavelet coefficients of random processes with long memory. These processes may be stationary or not and are obtained as the output of non–linear filter with Gaussian input. The wavelet coefficients that appear in the limit are random, typically non–Gaussian and belong to a Wiener chaos. They can be interpreted as wavelet coefficients of a generalized self-similar process.

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1. Introduction

Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a stationary Gaussian process with mean zero, unit variance and spectral density $f(\lambda), \lambda \in (-\pi, \pi]$ and thus covariance equal to

$$r(n) = \mathbb{E}(X_0 X_n) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\lambda} f(\lambda)\mathrm{d}\lambda \; .$$

The process $\{X_n\}_{n\in\mathbb{Z}}$ is said to have *short memory* or *short–range dependence* if $f(\lambda)$ is bounded around $\lambda = 0$ and *long memory* or *long–range dependence* if $f(\lambda) \to \infty$ as $\lambda \to 0$. We will suppose that $\{X_n\}_{n\in\mathbb{Z}}$ has long–memory with memory parameter $d > 0$, that is,

$$f(\lambda) \sim |\lambda|^{-2d} f^*(\lambda) \text{ as } \lambda \to 0$$

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where $f^*(\lambda)$ is a bounded spectral density which is continuous and positive at the origin. It is convenient to interpret this behavior as the result of a fractional integrating operation, whose transfer function reads $\lambda \mapsto (1 - \mathrm{e}^{-\mathrm{i}\lambda})^{-d}$. Hence we set

$$f(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{-2d} f^*(\lambda), \quad \lambda \in (-\pi, \pi] \; . \tag{1}$$

We relax the above assumptions in two ways :

1. Consider, instead of the Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ the non–Gaussian process $\{G(X_n)\}_{n\in\mathbb{Z}}$ where G is a non–linear filter such that $\mathbb{E}[G(X_n)] = 0$ and $\mathbb{E}[G(X_n)^2] < \infty$. The non–linear process $\{G(X_n)\}_{n\in\mathbb{Z}}$ is said to be subordinated to the Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$.

2. Drop the stationarity assumption by considering a process $\{Y_n\}_{n\in\mathbb{Z}}$ which becomes stationary when differenced $K \geq 0$ times.

We shall thus consider $\{Y_n\}_{n\in\mathbb{Z}}$ such that

$$\left(\Delta^K Y\right)_n = G(X_n), \quad n \in \mathbb{Z} \; ,$$

where $(\Delta Y)_n = Y_n - Y_{n-1}$ and where $\{X_n\}_{n\in\mathbb{Z}}$ is Gaussian with spectral density f satisfying (1).

Since $Y = \{Y_n\}_{n\in\mathbb{Z}}$ is random so will be its wavelet coefficients $\{W_{j,k},\ j \geq 0,\ k \in \mathbb{Z}\}$ which are defined below. Our goal is to find the distribution of the wavelet coefficients at large scales $j \to \infty$. This is an important step in developing methods for estimating the underlying long memory parameter d. The large scale behavior of the wavelet coefficients was studied in [1] in the case where there was no filter G, that is, when Y is a Gaussian process such that $\Delta^K Y = X$, and also in the case where Y is a non–Gaussian linear process (see [2]).

We obtain our random wavelet coefficients by using more general linear filters that those related to multiresolution analysis (MRA) (see for e.g. [3], [4]). In practice, however, the methods are best implemented using Mallat's algorithm and a MRA. Our filters are denoted h_j where j is the scale and we use a scaling factor $\gamma_j \uparrow \infty$ as $j \uparrow \infty$. In the case of a MRA, $\gamma_j = 2^j$ and h_j are generated by a (low pass) scaling filter and its corresponding quadratic (high pass) mirror filter. More generally one can use a scaling function φ and a mother wavelet ψ to generate the random wavelet coefficients by setting

$$W_{j,k} = \int_{\mathbb{R}} \psi_{j,k}(t) \left(\sum_{\ell\in\mathbb{Z}} \varphi(t-\ell) Y_\ell \right) \mathrm{d}t, \tag{2}$$

where $\psi_{j,k} = 2^{-j/2}\psi(2^{-j}t - k)$, $j \geq 0$. Observe that we use here the engineering convention that large values of j correspond to large scales and hence low frequencies. If φ and ψ have compact support then the corresponding filters h_j have finite support of size $O(2^j)$. For more details on related conditions on φ and ψ (see [1]).

The idea of using wavelets to estimate the long memory coefficient d goes back to Wornell and al. ([5]) and Flandrin ([6, 7, 8, 9]). See also Abry and al. ([10, 11]). Those methods are an alternative to the Fourier methods developed by Fox and Taqqu ([12]) and Robinson ([13, 14]. For a general comparison of Fourier and wavelet approach, see [15]. The case of the Rosenblatt process, which is the Hermite process of order $q = 2$, was studied by [16].

The paper is structured as follows. In Section 2, we introduce the wavelet filters. The processes are defined in Section 3 using integral representations and Section 4 presents the so–called Wiener chaos decomposition. The main result and its interpretations is given in Section 5. It is proved in Section 6. Auxiliary lemmas are presented and proved in Sections 7 and 8.

2. Assumptions on the wavelet filter

The wavelet transform of Y involves the application of a linear filter $h_j(\tau), \tau \in \mathbb{Z}$, at each scale $j \geq 0$. We shall characterize the filters h_j by their discrete Fourier transform :

$$\widehat{h}_j(\lambda) = \sum_{\tau\in\mathbb{Z}} h_j(\tau)\mathrm{e}^{-\mathrm{i}\lambda\tau},\ \lambda \in [-\pi, \pi]\ .$$

Assumptions on $\widehat{h}_j$ are stated below. The resulting wavelet coefficients are defined as

$$W_{j,k} = \sum_{\ell\in\mathbb{Z}} h_j(\gamma_j k - \ell)Y_\ell, \quad j \geq 0,\ k \in \mathbb{Z}\ ,$$

where $\gamma_j \uparrow \infty$ is a sequence of non–negative scale factors applied at scale j, for example $\gamma_j = 2^j$. We will assume that for any $m \in \mathbb{Z}$,

$$\lim_{j\to\infty} \frac{\gamma_{j+m}}{\gamma_j} = \overline{\gamma}_m > 0\ . \tag{3}$$

As noted, in this paper, we do not assume that the wavelet coefficients are orthogonal nor that they are generated by a multiresolution analysis. Our assumptions on the filters h_j are as follows :

a. Finite support: For each j, $\{h_j(\tau)\}_{\tau\in\mathbb{Z}}$ has finite support.

b. Uniform smoothness: There exists $M \geq K$, $\alpha > 1/2$ and $C > 0$ such that for all $j \geq 0$ and $\lambda \in [-\pi, \pi]$,

$$|\widehat{h}_j(\lambda)| \leq \frac{C\gamma_j^{1/2}|\gamma_j\lambda|^M}{(1 + \gamma_j|\lambda|)^{M+\alpha}}\ . \tag{4}$$

By 2π-periodicity of $\widehat{h}_j$ this inequality can be extended to $\lambda \in \mathbb{R}$ as

$$|\widehat{h}_j(\lambda)| \leq C\frac{\gamma_j^{1/2}|\gamma_j\{\lambda\}|^M}{(1 + \gamma_j|\{\lambda\}|)^{\alpha+M}}\ . \tag{5}$$

where $\{\lambda\}$ denotes the element of $(-\pi, \pi]$ such that $\lambda - \{\lambda\} \in 2\pi\mathbb{Z}$.

c. Asymptotic behavior: There exists some non identically zero function $\widehat{h}_\infty$ such that for any $\lambda \in \mathbb{R}$,

$$\lim_{j \to +\infty} (\gamma_j^{-1/2} \widehat{h}_j(\gamma_j^{-1}\lambda)) = \widehat{h}_\infty(\lambda) \ . \tag{6}$$

Observe that while $\widehat{h}_j$ is 2π-periodic, the function $\widehat{h}_\infty$ is a non-periodic function on $\mathbb{R}$ (this follows from (12) below). For the connection between these assumptions on h_j and corresponding assumptions on the scaling function φ and the mother wavelet ψ in the classical wavelet setting (2) (see [1]). In particular, in that case, one has $\widehat{h}_\infty = \widehat{\varphi}(0)\overline{\widehat{\psi}}$.

Our goal is to study the large scale behavior of the random wavelet coefficients

$$W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j(\gamma_j k - \ell) Y_\ell = \sum_{\ell \in \mathbb{Z}} h_j(\gamma_j k - \ell) \left(\Delta^{-K} G(X)\right)_\ell , \tag{7}$$

where we set symbolically $Y_\ell = \left(\Delta^{-K} G(X)\right)_\ell$ for $(\Delta^K Y)_\ell = G(X_\ell)$.

By Assumption (4), h_j has null moments up to order $M-1$, that is, for any $m \in \{0, \cdots, M-1\}$,

$$\sum_{\ell \in \mathbb{Z}} h_j(\ell) \ell^m = 0 \ . \tag{8}$$

Therefore, since $M \geq K$, $\widehat{h}_j$ can be expressed as

$$\widehat{h}_j(\lambda) = (1 - \mathrm{e}^{-\mathrm{i}\lambda})^K \widehat{h}_j^{(K)}(\lambda), \tag{9}$$

where $\widehat{h}_j^{(K)}$ is also a trigonometric polynomial of the form

$$\widehat{h}_j^{(K)}(\lambda) = \sum_{\tau \in \mathbb{Z}} h_j^{(K)}(\tau) \mathrm{e}^{-\mathrm{i}\lambda\tau}, \tag{10}$$

since $h_j^{(K)}$ has finite support for any j. Then we obtain another way of expressing $W_{j,k}$, namely,

$$W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) G(X_\ell) \ . \tag{11}$$

We have thus incorporated the linear filter Δ^{-K} in (7) into the filter h_j and denoted the new filter $h_j^{(K)}$.

Remarks

1. Since $\{G(X_\ell), \ell \in \mathbb{Z}\}$ is stationary, it follows from (11) that $\{W_{j,k}, k \in \mathbb{Z}\}$ is stationary for each scale j.

2. Observe that $\Delta^K Y$ is centered by definition. However, by (8), the definition of $W_{j,k}$ only depends on $\Delta^M Y$. In particular, provided that $M \geq K+1$, its value is not modified if a constant is added to $\Delta^K Y$, whenever $M \geq K+1$.

3. Assumptions (4) and (6) imply that for any $\lambda \in \mathbb{R}$,

$$|\widehat{h}_\infty(\lambda)| \le C \frac{|\lambda|^M}{(1+|\lambda|)^{\alpha+M}} . \tag{12}$$

Hence $\widehat{h}_\infty \in L^2(\mathbb{R})$ since $\alpha > 1/2$.

4. The Fourier transform of f,

$$\mathfrak{F}(f)(\xi) = \int_{\mathbb{R}^q} f(t) \mathrm{e}^{-\mathrm{i} t^T \xi} \, \mathrm{d}^q t, \quad \xi \in \mathbb{R}^q , \tag{13}$$

is defined for any $f \in L^2(\mathbb{R}^q, \mathbb{C})$. We let h_∞ be the $L^2(\mathbb{R})$ function such that $\widehat{h}_\infty = \mathfrak{F}[h_\infty]$.

3. Integral representations

It is convenient to use an integral representation in the spectral domain to represent the random processes (see for example [17, 18]). The stationary Gaussian process $\{X_k, k \in \mathbb{Z}\}$ with spectral density (1) can be written as

$$X_\ell = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\lambda\ell} f^{1/2}(\lambda) \mathrm{d}\widehat{W}(\lambda) = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\lambda\ell} f^{*1/2}(\lambda)}{|1-\mathrm{e}^{-\mathrm{i}\lambda}|^d} \mathrm{d}\widehat{W}(\lambda), \quad \ell \in \mathbb{N} . \tag{14}$$

This is a special case of

$$\widehat{I}(g) = \int_{\mathbb{R}} g(x) \mathrm{d}\widehat{W}(x), \tag{15}$$

where $\widehat{W}(\cdot)$ is a complex–valued Gaussian random measure satisfying

$$\begin{aligned}
\mathbb{E}(\widehat{W}(A)) &= 0 \quad \text{for every Borel set } A \text{ in } \mathbb{R} , && (16)\\
\mathbb{E}(\widehat{W}(A)\overline{\widehat{W}(B)}) &= |A \cap B| \text{ for every Borel sets } A \text{ and } B \text{ in } \mathbb{R} , && (17)\\
\sum_{j=1}^{n} \widehat{W}(A_j) &= \widehat{W}(\bigcup_{j=1}^{n} A_j) \text{ if } A_1, \cdots, A_n \text{ are disjoint Borel sets in } \mathbb{R} , && (18)\\
\widehat{W}(A) &= \overline{\widehat{W}(-A)} \quad \text{for every Borel set } A \text{ in } \mathbb{R} . && (19)
\end{aligned}$$

The integral (15) is defined for any function $g \in L^2(\mathbb{R})$ and one has the isometry

$$\mathbb{E}(|\widehat{I}(g)|^2) = \int_{\mathbb{R}} |g(x)|^2 \mathrm{d}x .$$

The integral $\widehat{I}(g)$, moreover, is real–valued if

$$g(x) = \overline{g(-x)} .$$

We shall also consider multiple Itô–Wiener integrals

$$\widehat{I}_q(g) = \int_{\mathbb{R}^q}'' g(\lambda_1, \cdots, \lambda_q) \mathrm{d}\widehat{W}(\lambda_1) \cdots \mathrm{d}\widehat{W}(\lambda_q)$$

where the double prime indicates that one does not integrate on hyperdiagonals $\lambda_i = \pm\lambda_j, i \neq j$. The integrals $\widehat{I}_q(g)$ are handy because we will be able to expand our non–linear functions $G(X_k)$ introduced in Section 1 in multiple integrals of this type.

These multiples integrals are defined as follows. Denote by $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$ the space of complex valued functions defined on $\mathbb{R}^q$ satisfying

$$g(-x_1, \cdots, -x_q) = \overline{g(x_1, \cdots, x_q)} \text{ for } (x_1, \cdots, x_q) \in \mathbb{R}^q , \tag{20}$$

$$\|g\|_{L^2}^2 := \int_{\mathbb{R}^q} |g(x_1, \cdots, x_q)|^2 \, \mathrm{d}x_1 \cdots \mathrm{d}x_q < \infty . \tag{21}$$

Let $\tilde{L}^2(\mathbb{R}^q, \mathbb{C})$ denote the set of functions in $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$ that are symmetric in the sense that $g = \tilde{g}$ where $\tilde{g}(x_1, \cdots, x_q) = 1/q! \sum_\sigma g(x_{\sigma(1)}, \cdots, x_{\sigma(q)})$, where the sum is over all permutations of $\{1, \ldots, q\}$. One defines now the multiple integral with respect to the spectral measure $\widehat{W}$ by a density argument. For a step function of the form

$$g = \sum_{j_\ell = \pm 1, \cdots, \pm N} c_{j_1, \cdots, j_n} 1_{\Delta_{j_1}} \times \cdots \times 1_{\Delta_{j_n}}$$

where the c's are real–valued, $\Delta_{j_\ell} = -\Delta_{-j_\ell}$ and $\Delta_{j_\ell} \cap \Delta_{j_m} = \emptyset$ if $\ell \neq m$, one sets

$$\widehat{I}_q(g) = \sum_{j_\ell = \pm 1, \cdots, \pm N}'' c_{j_1, \cdots, j_n} \widehat{W}(\Delta_{j_1}) \cdots \widehat{W}(\Delta_{j_n}) . \tag{22}$$

Here, $\sum''$ indicates that one does not sum over the hyperdiagonals, that is, when $j_\ell = \pm j_m$ for $\ell \neq m$. The integral $\widehat{I}_q$ verifies that

$$\mathbb{E}(\widehat{I}_q(g_1)\widehat{I}_{q'}(g_2)) = \begin{cases} q!\langle g_1, g_2 \rangle_{L^2}, & \text{if } q = q' \\ 0, & \text{if } q \neq q'. \end{cases} \tag{23}$$

Observe, moreover, that for every step function g with q variables as above

$$\widehat{I}_q(g) = \widehat{I}_q(\tilde{g}).$$

Since the set of step functions is dense in $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$, one can extend $\widehat{I}_q$ to an isometry from $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$ to $L^2(\Omega)$ and the above properties hold true for this extension.

Remark. Property (20) of the function f in $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$ together with Property (19) of $\widehat{W}$ ensure that $\widehat{I}_q(f)$ is a real–valued random variable.

4. Wiener Chaos

Our results are based on the expansion of the function G, introduced in Section 1, in Hermite polynomials. The Hermite polynomials are

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\left(e^{-\frac{x^2}{2}}\right) ,$$

in particular, $H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1$. If X is a normal random variable with mean 0 and variance 1, then

$$\mathbb{E}(H_q(X)H_{q'}(X)) = \int_{\mathbb{R}} H_q(x)H_{q'}(x)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\mathrm{d}x = q!\delta_{q,q'} .$$

Moreover,

$$G(X) = \sum_{q=1}^{+\infty} \frac{c_q}{q!} H_q(X) , \tag{24}$$

where the convergence is in $L^2(\Omega)$ and where

$$c_q = \mathbb{E}(G(X)H_q(X)) . \tag{25}$$

The expansion (24) is called a Wiener chaos expansion with each term in the chaos expansion living in a different chaos. The expansion (24) starts at $q = 1$, since

$$c_0 = \mathbb{E}(G(X)H_0(X)) = \mathbb{E}(G(X)) = 0 ,$$

by assumption. The condition $\mathbb{E}(G(X)^2) < \infty$ implies

$$\sum_{q=1}^{+\infty} \frac{c_q^2}{q!} < \infty . \tag{26}$$

Hermite polynomials are related to multiple integrals as follows : if $X = \int_{\mathbb{R}} g(x)\mathrm{d}\widehat{W}(x)$ with $\mathbb{E}(X^2) = \int_{\mathbb{R}} |g(x)|^2\,\mathrm{d}x = 1$ and $g(x) = \overline{g(-x)}$ so that X has unit variance and is real–valued, then

$$H_q(X) = \widehat{I}_q(g^{\otimes q}) = \int_{\mathbb{R}^q}'' g(x_1)\cdots g(x_q)\mathrm{d}\widehat{W}(x_1)\cdots\mathrm{d}\widehat{W}(x_q) . \tag{27}$$

The expansion (24) of G induces a corresponding expansion of the wavelet coefficients $W_{j,k}$, namely,

$$W_{j,k} = \sum_{q=1}^{+\infty} \frac{c_q}{q!} W_{j,k}^{(q)} , \tag{28}$$

where by (11) one has

$$W_{j,k}^{(q)} = \sum_{\ell\in\mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell)H_q(X_\ell) . \tag{29}$$

The Gaussian sequence $\{X_n\}_{n\in\mathbb{Z}}$ is long–range dependent because its spectrum at low frequencies behaves like $|\lambda|^{-2d}$ with $d>0$ and hence explodes at $\lambda=0$. What about the processes $\{H_q(X_\ell)\}_\ell$ for $q\geq 2$? What is the behavior of the spectrum at low frequencies? Does it explodes at $\lambda=0$? The answer depends on the respective values of q and d. Let us define

$$q_c=\max\{q\in\mathbb{N}\ :\ q<1/(1-2d)\}\ , \tag{30}$$

and

$$d(q)=qd+(1-q)/2\ . \tag{31}$$

One has

$$d(q)>0\quad \text{if } q\leq q_c,\quad \text{that is if } q<1/(1-2d)\ . \tag{32}$$

The following result shows that the spectral density of $\{H_q(X_\ell)\}_{\ell\in\mathbb{Z}}$ has a different behavior at zero frequency depending on whether $q\leq q_c$ or $q>q_c$. It is long–range dependent when $q\leq q_c$ and short–range dependent when $q>q_c$. We first give a definition.

Definition 4.1. *The convolution of two locally integrable (2π)-periodic functions g_1 and g_2 is defined as*

$$(g_1\star g_2)(\lambda)=\int_{-\pi}^{\pi}g_1(u)g_2(\lambda-u)\mathrm{d}u\ . \tag{33}$$

Moreover the q times self-convolution of g is denoted by $g^{(\star q)}$.

Lemma 4.1. *Let q be a positive integer. The spectral density of $\{H_q(X_\ell)\}_{\ell\in\mathbb{Z}}$ is*

$$q!f^{(\star q)}=q!(f\star\cdots\star f)\ ,$$

where the spectral density f of $\{X_\ell\}_{\ell\in\mathbb{Z}}$ is given in (1). Moreover the following holds :

- *(i) If $q\leq q_c$, then $\lambda^{2d(q)}f^{(\star q)}(\lambda)$ is bounded on $\lambda\in(0,\pi)$ and converges to a positive number as $\lambda\downarrow 0$.*
- *(ii) If $q>q_c$, then $f^{(\star q)}(\lambda)$ is bounded on $\lambda\in(0,\pi)$ and converges to a positive number as $\lambda\downarrow 0$.*

Hence if $q\leq q_c$, $\{H_q(X_\ell)\}_\ell$ has long memory with parameter $d(q)>0$ whereas if $q>q_c$, $\{H_q(X)\}_\ell$ has a short–memory behavior.

Proof. By definition of H_q and since X has unit variance by assumption, we have

$$\mathbb{E}(H_q(X_\ell)H_q(X_{\ell+m}))=q!\left(\int_{-\pi}^{\pi}f(\lambda)\mathrm{e}^{\mathrm{i}\lambda m}\mathrm{d}\lambda\right)^q\ .$$

Using the fact that, for any two locally integrable (2π)-periodic functions g_1 and g_2, one has

$$\int_{-\pi}^{\pi}(g_1\star g_2)(\lambda)\mathrm{e}^{\mathrm{i}\lambda m}\mathrm{d}\lambda=\int_{-\pi}^{\pi}g_1(u)\mathrm{e}^{\mathrm{i}um}\mathrm{d}u\times\int_{-\pi}^{\pi}g_2(v)\mathrm{e}^{\mathrm{i}vm}\mathrm{d}v\ ,$$

we obtain that the spectral density of $\{H_q(X_\ell)\}_\ell$ is $q! f^{(\star q)}$.

The properties of $f^{(\star q)}$ stated in Lemma 4.1 are proved by induction on q using Lemma 8.2. Observe indeed that if $\beta_1 = d(q)$ and $\beta_2 = 2d$, then

$$\beta_1 + \beta_2 - 1 = 2d(q) + 2d - 1 = (2dq + 1 - q) + 2d - 1 = 2(q+1)d - (q+1) + 1 = 2d(q+1) \; .$$

$\square$

Now, consider the expansion of $\Delta^K Y_\ell = G(X_\ell) = \sum_{q=q_0}^{+\infty} (c_q/q!) H_q(X_\ell)$, where

$$q_0 = \min\{q \geq 1,\, c_q \neq 0\} \; . \tag{34}$$

The exponent q_0 is called the *Hermite rank* of $\Delta^K Y$.

In the following, we always assume that at least one summand of $\Delta^K Y_\ell$ has long memory, that is, in view of Lemma 4.1,

$$q_0 \leq q_c \; . \tag{35}$$

5. The result and its interpretations

In this section we describe the limit in distribution of the wavelet coefficients $\{W_{j+m,k}\}_{m,k}$ as $j \to \infty$, adequately normalized, and we interpret the limit. Recall that $W_{j+m,k}$ involves a sum of chaoses of all order. In the limit, however, only the order q_0 will prevail. The convergence of finite–dimensional distributions is denoted by $\overset{\text{fidi}}{\to}$.

Theorem 5.1. *As $j \to \infty$, we have*

$$\left\{ \gamma_j^{-(d(q_0)+K)} W_{j+m,k},\, m,k \in \mathbb{Z} \right\} \overset{fidi}{\to} c_{q_0} \left(f^*(0)\right)^{q_0/2} \left\{ Y_{m,k}^{(q_0,K)},\, m,k \in \mathbb{Z} \right\} \; , \tag{36}$$

where for every positive integer q,

$$Y_{m,k}^{(q,K)} = (\overline{\gamma}_m)^{1/2} \int_{\mathbb{R}^q}'' \frac{\mathrm{e}^{\mathrm{i}k\overline{\gamma}_m(\zeta_1+\cdots+\zeta_q)}}{(\mathrm{i}(\zeta_1+\cdots+\zeta_q))^K} \frac{\widehat{h}_\infty(\overline{\gamma}_m(\zeta_1+\cdots+\zeta_q))}{|\zeta_1|^d \cdots |\zeta_q|^d} \,\mathrm{d}\widehat{W}(\zeta_1)\cdots \mathrm{d}\widehat{W}(\zeta_q) \; . \tag{37}$$

This Theorem is proved in Section 6.

Interpretation of the limit.

The limit distribution can be interpreted as the wavelet coefficients of a generalized Hermite process defined below, based on the wavelet family

$$\left\{ h_{\infty,m,k}(t) = \overline{\gamma}_m^{-1/2} h_\infty(-\overline{\gamma}_m^{-1} t + k),\, m,k \in \mathbb{Z} \right\} \; . \tag{38}$$

This wavelet family is the natural one to consider because the Fourier transform $\widehat{h}_\infty(\lambda)$ is the rescaled limit of the original $\widehat{h}_j(\lambda)$ as indicated in (6).

A generalized process is indexed not by time but by functions. The generalized Hermite processes for any order q in $\{1, \ldots, q_c\}$ are defined as follows :

Definition 5.1. *Let $0 < d < 1/2$ and let q be a positive integer such that $0 < q < 1/(1-2d)$ and $K \geq 0$. Define the set of functions*

$$\mathcal{S}_{q,d}^{(K)} = \left\{ \theta \ , \int_{\mathbb{R}} \left| \widehat{\theta}(\xi) \right|^2 \ |\xi|^{q-1-2dq-2K} \mathrm{d}\xi < \infty \right\} \ ,$$

where $\widehat{\theta} = \mathfrak{F}[\theta]$. The generalized random process $Z_{q,d}^{(K)}$ is indexed by functions $\theta \in \mathcal{S}_{q,d}^{(K)}$ and is defined as

$$Z_{q,d}^{(K)}(\theta) = \int_{\mathbb{R}^q}'' \frac{\overline{\widehat{\theta}(u_1 + \cdots + u_q)}}{(\mathrm{i}(u_1 + \cdots + u_q))^K |u_1 \cdots u_q|^d} \, \mathrm{d}\widehat{W}(u_1) \cdots \mathrm{d}\widehat{W}(u_q) \ , \tag{39}$$

where $\widehat{\theta} = \mathfrak{F}[\theta]$ as defined in (13).

Now fix $(m, k) \in \mathbb{Z}^2$ and choose a function $h_{\infty,m,k}(t), t \in \mathbb{R}$ as in (38), so that

$$\mathfrak{F}[h_{\infty,m,k}](\xi) = \mathfrak{F}[\overline{\gamma}_m^{-1/2} h_\infty(-\overline{\gamma}_m^{-1} t + k)](\xi) = (\overline{\gamma}_m)^{1/2} \ \mathrm{e}^{-\mathrm{i}\overline{\gamma}_m \xi} \ \overline{\widehat{h}_\infty(\overline{\gamma}_m \xi)} \ . \tag{40}$$

Lemma 5.1. *The conditions on d and q in Definition 5.1 ensures the existence of $Z_{q,d}^{(K)}(\theta)$. In particular,*

$$h_{\infty,m,k} \in \mathcal{S}_{q,d}^{(K)} \ \text{for all } K \in \{0, \ldots, M\} \ ,$$

and hence $Z_{q,d}^{(K)}(h_{\infty,m,k})$ is well-defined.

This Lemma is proved in Section 7.

By setting in (39), $\theta = h_{\infty,m,k}$, defined in (40), we obtain for all $(m, k) \in \mathbb{Z}^2$,

$$Y_{m,k}^{(q,K)} = Z_{q,d}^{(K)}(h_{\infty,m,k}) \ .$$

Hence the right-hand side of (36) are the wavelet coefficients of the generalized process $Z_{q,d}^{(K)}$ with respect to the wavelet family $\{h_{\infty,m,k}, \ m, k \in \mathbb{Z}\}$.

In the special case $q = 1$ (Gaussian case), this result corresponds to that of Theorem 1(b) and Remark 5 in [1], obtained in the case where $\gamma_j = 2^j$. In this special case, we have $Z_{1,d}^{(K)} = B_{(d+K)}$, where $B_{(d)}$ is the centered generalized Gaussian process such that for all $\theta_1, \theta_2 \in \mathcal{S}_{1,d}^{(0)}$,

$$\mathrm{Cov}(B_{(d)}(\theta_1), B_{(d)}(\theta_2)) = \int_{\mathbb{R}} |\lambda|^{-2d} \widehat{\theta}_1(\lambda) \overline{\widehat{\theta}_2(\lambda)} \, \mathrm{d}\lambda \ .$$

It is interesting to observe that, under additional assumptions on θ, for $K \geq 1$, $Z_{q,d}^{(K)}(\theta)$ can also be defined by

$$Z_{q,d}^{(K)}(\theta) = \int_{\mathbb{R}} \tilde{Z}_{q,d}^{(K)}(t) \overline{\theta(t)} \, \mathrm{d}t \ , \tag{41}$$

where $\{\tilde{Z}_{q,d}^{(K)}(t), \ t \in \mathbb{R}\}$ denotes a measurable continuous time process defined by

$$\tilde{Z}_{q,d}^{(K)}(t) = \int_{\mathbb{R}^q}'' \frac{\mathrm{e}^{\mathrm{i}(u_1 + \cdots + u_q) t} - \sum_{\ell=0}^{K-1} \frac{(\mathrm{i}(u_1 + \cdots + u_q) t)^\ell}{\ell!}}{(\mathrm{i}(u_1 + \cdots + u_q))^K |u_1 \cdots u_q|^d} \, \mathrm{d}\widehat{W}(u_1) \cdots \mathrm{d}\widehat{W}(u_q), \ t \in \mathbb{R} \ . \tag{42}$$

If, in (41) we set $K = 1$, we recover the usual Hermite process as defined in [19] which has stationary increments. The process $\tilde{Z}^{(K)}_{q,d}(t)$ can be regarded as the Hermite process $\tilde{Z}^{(1)}_{q,d}(t)$ integrated $K-1$ times. In the special case where $K = q = 1$, we recover the Fractional Brownian Motion $\{B_H(t)\}_{t\in\mathbb{R}}$ with Hurst index $H = d + 1/2 \in (1/2, 1)$.

In the case $K = 0$ we cannot define a random process $Z^{(0)}_{q,d}(t)$ as in (42). The case $K = 0$ would correspond to the derivative of the Hermite process $\tilde{Z}^{(1)}_{q,d}(t)$ but the Hermite process is not differentiable and thus the process $\tilde{Z}^{(0)}_{q,d}(t), t \in \mathbb{R}$ is not defined. When $K = 0$ one can only consider the generalized process $\tilde{Z}^{(0)}_{q,d}(\theta)$. Relation (42) can be viewed as resulting from (39) and (41) by interverting formally the integral signs.

We now state sufficient conditions on θ for (41) to hold.

Lemma 5.2. *Let q be a positive integer such that $0 < q < 1/(1-2d)$ and $K \geq 1$. Suppose that $\theta \in \mathcal{S}^{(K)}_{q,\delta}$ is complex valued with at least K vanishing moments, that is,*

$$\int_{\mathbb{R}} \theta(t)\, t^{\ell}\, \mathrm{d}t = 0 \quad \textit{for all} \quad \ell = 0, 1, \ldots, K-1 \,. \tag{43}$$

Suppose moreover that

$$\int_{\mathbb{R}} |\theta(t)|\, |t|^{K+(d-1/2)q}\, \mathrm{d}t < \infty \,. \tag{44}$$

Then Relation (41) holds.

This lemma is proved in Section 7.

If, for example, the h_j are derived from a compactly supported multiresolution analysis then h_∞ will have compact support and so $h_{\infty,m,k}$ will satisfy (44). In this case, the limits $Y^{(q,K)}_{m,k}$ in Theorem 5.1 can therefore be interpreted, for $m, k \in \mathbb{Z}$ as the wavelet coefficients of the process $Z^{(K)}_{q,d}$ belonging to the q–th chaos. This interpretation is a useful one even when the technical assumption (44) is not satisfied.

Self-similarity.

The processes $Z^{(K)}_{q,d}$ and $\tilde{Z}^{(K)}_{q,d}$ are self-similar. Self-similarity can be defined for processes indexed by $t \in \mathbb{R}$ as well as for generalized processes indexed by functions θ belonging to some suitable space $\mathcal{S}$, for example the space $\mathcal{S}^{(K)}_{q,\delta}$ defined above.

A process $\{Z(t),\, t \in \mathbb{R}\}$ is said to be *self-similar with parameter* $H > 0$ if for any $a > 0$,

$$\{a^H\, Z(t/a),\, t \in \mathbb{R}\} \overset{\text{fidi}}{=} \{Z(t),\, t \in \mathbb{R}\} \,,$$

where the equality holds in the sense of finite-dimensional distributions. A generalized process $\{Z(\theta),\, \theta \in \mathcal{S}\}$ is said to be *self-similar with parameter* $H > 0$ if for any $a > 0$ and $\theta \in \mathcal{S}$,

$$Z(\theta^{a,H}) \overset{\mathrm{d}}{=} Z(\theta) \,,$$

where $\theta^{a,H}(u) = a^{-H}\theta(u/a)$ (see [17], Page 5). Here $\mathcal{S}$ is assumed to contain both $\theta^{a,H}$ and θ.

Observe that the process $\{\tilde{Z}^{(K)}_{q,d}(t),\, t \in \mathbb{R}\}$, with $K \geq 1$ is self-similar with parameter

$$H = K + qd - q/2 = (K-1) + (d(q) + 1/2) \; . \tag{45}$$

As noted above $\tilde{Z}^{(K)}_{q,d}$ can be regarded as $\tilde{Z}^{(1)}_{q,d}$ integrated $K-1$ times.

The generalized process $\{Z^{(K)}_{q,d}(\theta),\, \theta \in \mathcal{S}^{(K)}_{q,\delta}\}$, which is defined in (39) with $K \geq 0$, is self-similar with the same value of H as in (45), but this time the formula is also valid for $K = 0$.

In particular, the Hermite process ($K = 1$) is self-similar with $H = d(q)+1/2 \in (1/2, 1)$ and the generalized process $Z^{(0)}_{q,d}(\theta)$ with $K = 0$ is self-similar with $H = d(q) - 1/2 \in (-1/2, 0)$.

Interpretation of the result.

In view of the preceding discussion, the wavelet coefficients of the subordinated process Y behave at large scales ($\gamma_j \to \infty$) as those of a self-similar process $Z^{(K)}_{q,d}$ living in the chaos of order q_0 (the Hermite rank of G) and with self-similar parameter $K + d(q_0) - 1/2$.

6. Proof of Theorem 5.1

Notation. It will be convenient to use the following notation. We denote by Σ_q, $q \geq 1$, the $\mathbb{C}^q \to \mathbb{C}$ function defined, for all $y = (y_1, \dots, y_q)$ by

$$\Sigma_q(y) = \sum_{i=1}^{q} y_i \; . \tag{46}$$

With this notation $Y^{(q,K)}_{m,k}$ in Theorem 5.1 can be expressed as

$$Y^{(q,K)}_{m,k} = (\overline{\gamma}_m)^{1/2} \int''_{\mathbb{R}^q} \frac{\exp \circ \Sigma_q(\mathrm{i}k\overline{\gamma}_m\zeta)}{(\Sigma_q(\mathrm{i}\zeta))^K} \cdot \frac{\widehat{h}_\infty \circ \Sigma_q(\overline{\gamma}_m\zeta)}{|\zeta_1|^d \cdots |\zeta_q|^d} \, \mathrm{d}\widehat{W}(\zeta_1) \cdots \mathrm{d}\widehat{W}(\zeta_q) \; .$$

where $\circ$ denotes the composition of functions.

We will separate the Wiener chaos expansion (28) of $W_{j,k}$ into two terms depending on the position of q with respect to q_c. The first term includes only the q's for which $H_q(x)$ exhibits long–range dependence (LD), that is,

$$W^{(LD)}_{j,k} = \sum_{q=0}^{q_c} \frac{c_q}{q!} W^{(q)}_{j,k} \; , \tag{47}$$

and the second term includes the terms which exhibit short–range dependence (SD)

$$W_{j,k}^{(SD)} = \sum_{q=q_c+1}^{\infty} \frac{c_q}{q!} W_{j,k}^{(q)} . \tag{48}$$

Using Representation (14) and (27) since X has unit variance, one has for any $\ell \in \mathbb{Z}$,

$$\begin{aligned} H_q(X_\ell) &= H_q\left(\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\xi\ell} f^{1/2}(\xi) \mathrm{d}\widehat{W}(\xi)\right) \\ &= \int_{(-\pi,\pi]^q}'' \exp \circ \Sigma_q(\mathrm{i}\ell\xi) \times \left(f^{\otimes q}(\xi)\right)^{1/2} \, \mathrm{d}\widehat{W}(\xi_1)\cdots \mathrm{d}\widehat{W}(\xi_q) . \end{aligned}$$

Then by (29),(10) and (9), we have

$$\begin{aligned} W_{j,k}^{(q)} &= \sum_{\ell\in\mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) H_q(X_\ell) \\ &= \sum_{\ell\in\mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) \int_{(-\pi,\pi]^q}'' \exp \circ \Sigma_q(\mathrm{i}\ell\xi) \times \left(f^{\otimes q}(\xi)\right)^{1/2} \, \mathrm{d}\widehat{W}(\xi_1)\cdots \mathrm{d}\widehat{W}(\xi_q) \\ &= \int_{(-\pi,\pi]^q}'' \left(\sum_{\ell\in\mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) \exp \circ \Sigma_q(\mathrm{i}\ell\xi)\right) \left(f^{\otimes q}(\xi)\right)^{1/2} \, \mathrm{d}\widehat{W}(\xi_1)\cdots \mathrm{d}\widehat{W}(\xi_q) \\ &= \int_{(-\pi,\pi]^q}'' \mathrm{e}^{\Sigma_q(\mathrm{i}\gamma_j k\xi)} \left(\sum_{m\in\mathbb{Z}} h_j^{(K)}(m) \exp \circ \Sigma_q(-\mathrm{i}m\xi)\right) \left(f^{\otimes q}(\xi)\right)^{1/2} \, \mathrm{d}\widehat{W}(\xi_1)\cdots \mathrm{d}\widehat{W}(\xi_q) \\ &= \int_{(-\pi,\pi]^q}'' \mathrm{e}^{\Sigma_q(\mathrm{i}\gamma_j k\xi)} \left(\widehat{h}_j^{(K)} \circ \Sigma_q(\xi)\right) \left(f^{\otimes q}(\xi)\right)^{1/2} \, \mathrm{d}\widehat{W}(\xi_1)\cdots \mathrm{d}\widehat{W}(\xi_q) . \end{aligned}$$

Then

$$W_{j,k}^{(q)} = \widehat{I}_q(f_{j,k}^{(q)}) , \tag{49}$$

with

$$f_{j,k}^{(q)}(\xi) = (\exp \circ \Sigma_q(\mathrm{i}k\gamma_j\xi)) \left(\widehat{h}_j^{(K)} \circ \Sigma_q(\xi)\right) \left(f^{\otimes q}(\xi)\right)^{1/2} \times \mathbb{1}_{(-\pi,\pi)}^{\otimes q}(\xi) ,$$

where $\xi = (\xi_1, \cdots, \xi_q)$ and $f^{\otimes q}(\xi) = f(\xi_1)\cdots f(\xi_q)$.

The two following results provide the asymptotic behavior of each term of the sum in (47) and of $W_{j,k}^{(SD)}$, respectively. They are proved in Sections 6.1 and 6.2, respectively. The first result concerns the terms with long memory, that is, with $q \le q_c$. The second result concerns the terms with short memory for which $q > q_c$.

Proposition 6.1. *Suppose that $q \in \{1, \ldots, q_c\}$. Then, as $j \to \infty$,*

$$\left(\gamma_j^{-(d(q)+K)} W_{j+m,k}^{(q)},\, m, k \in \mathbb{Z}\right) \xrightarrow{\text{fidi}} \left((f^*(0))^{q/2}\, Y_{m,k}^{(q,K)},\, m, k \in \mathbb{Z}\right) , \tag{50}$$

where $Y_{m,k}^{(q,K)}$ is given by (37).

Proposition 6.2. *We have, for any $k \in \mathbb{Z}$, as $j \to \infty$,*

$$W^{(SD)}_{j+m,k} = O_P(\gamma_j^K) \ . \tag{51}$$

It follows from Proposition 6.1 that the dominating term in (47) is given by the chaos of order $q = q_0$. Now, since $d(q_0) > 0$ by (32), we get from Proposition 6.2 that, for all (k, m), as $j \to \infty$,

$$W^{(SD)}_{j+m,k} = o_p(\gamma_j^{d(q_0)+K}) \ .$$

This concludes the proof of Theorem 5.1.

6.1. Proof of Proposition 6.1

We first express the distribution of $\{W^{(q)}_{j+m,k},\ m, k \in \mathbb{Z}\}$ as a finite sum of stochastic integrals and then show that each integral converges in $L^2(\Omega)$.

Lemma 6.1. *Let $q \in \mathbb{N}^*$. For any j*

$$W^{(q)}_{j+m,k} \overset{(fidi)}{=} \sum_{s=-[q/2]}^{[q/2]} W^{(j,q,s)}_{m,k} \ , \tag{52}$$

where $[a]$ denotes the integer part of a, and for any $q \in \mathbb{N}^*$, $s \in \mathbb{Z}$,

$$W^{(j,q,s)}_{m,k} = \int''_{\zeta \in \mathbb{R}^q} \mathbb{1}_{\Gamma^{(q,s)}}(\gamma_j^{-1}\zeta) f_{m,k}(\zeta; j, q) \, \mathrm{d}\widehat{W}(\zeta_1) \cdots \mathrm{d}\widehat{W}(\zeta_q) \ , \tag{53}$$

where $f_{m,k}(\zeta; j, q)$ is defined by (setting $\xi = \gamma_j^{-1}\zeta$)

$$f_{m,k}(\gamma_j\xi; j, q) = \gamma_j^{-q/2} \frac{\exp \circ \Sigma_q(\mathrm{i}\gamma_{j+m}k\xi) \times \widehat{h}_{j+m} \circ \Sigma_q(\xi)}{\{1 - \exp \circ \Sigma_q(-\mathrm{i}\xi)\}^K} \left(f^{\otimes q}(\xi)\right)^{1/2} . \tag{54}$$

and where

$$\Gamma^{(q,s)} = \left\{ \xi \in (-\pi, \pi]^q,\ -\pi + 2s\pi < \sum_{i=1}^{q} \xi_i \le \pi + 2s\pi \right\} \ . \tag{55}$$

Proof. Using (49), with j replaced by $j + m$, and (9), we get

$$W^{(q)}_{j+m,k} = \int''_{(-\pi,\pi]^q} \exp \circ \Sigma_q(\mathrm{i}\gamma_{j+m}k\xi) \frac{\widehat{h}_{j+m} \circ \Sigma_q(\xi)}{\{1 - \exp \circ \Sigma_q(\mathrm{i}\xi\}^K} \left(f^{\otimes q}(\xi)\right)^{1/2} \ \mathrm{d}\widehat{W}(\xi_1) \cdots \mathrm{d}\widehat{W}(\xi_q) \ .$$

By (54), we thus get

$$\begin{aligned} W^{(q)}_{j+m,k} &= \int''_{\xi \in (-\pi,\pi]^q} \gamma_j^{q/2} f_{m,k}(\gamma_j\xi; j, q) \, \mathrm{d}\widehat{W}(\xi_1) \cdots \mathrm{d}\widehat{W}(\xi_q) \\ &\overset{(\mathrm{fidi})}{=} \int''_{\zeta \in (-\gamma_j\pi, \gamma_j\pi]^q} f_{m,k}(\zeta; j, q) \, \mathrm{d}\widehat{W}(\zeta_1) \cdots \mathrm{d}\widehat{W}(\zeta_q) \ , \end{aligned} \tag{56}$$

where we set $\zeta = \gamma_j \xi$ (see Theorem 4.4 in [17]). Observe that for all $\zeta \in (-\gamma_j\pi, \gamma_j\pi]^q$,

$$-\pi\gamma_j - 2[q/2]\pi\gamma_j \leq -q\gamma_j\pi \leq \sum_{i=1}^q \zeta_i \leq q\gamma_j\pi \leq \pi\gamma_j + 2[q/2]\pi\gamma_j \ .$$

The result follows by using that for any $\zeta \in (-\gamma_j\pi, \gamma_j\pi]^q$, there is a unique $s = -[q/2], \ldots, [q/2]$ such that $\zeta/\gamma_j \in \Gamma^{(q,s)}$. □

Proof of Proposition 6.1. In view of Lemma 6.1, we shall look at the $L^2(\Omega)$ convergence of the normalized $W_{m,k}^{(j,q,s)}$ at each value of s. Proposition 6.1 will follow from the following convergence results, valid for all fixed $m, k \in \mathbb{Z}$ as $j \to \infty$. For $s = 0$,

$$\gamma_j^{-(d(q)+K)} W_{m,k}^{(j,q,0)} \xrightarrow{L^2} (f^*(0))^{q/2}\, Y_{m,k}^{(q,K)} \ , \tag{57}$$

whereas for other values of s, namely for all $s \in \{-[q/2], \ldots, -1, 1, \ldots, [q/2]\}$,

$$\gamma_j^{-(d(q)+K)} W_{m,k}^{(j,q,s)} \xrightarrow{L^2} 0 \ , \tag{58}$$

where $d(q)$ is defined in (31).

We now prove these convergence using the representation (53). By (1) and $|1 - \mathrm{e}^{\mathrm{i}\lambda}| \geq 2|\lambda|/\pi$ on $\lambda \in (-\pi, \pi)$, we have that

$$f(\lambda) \leq \left(\frac{\pi}{2}\right)^{-2d} \|f^*\|_\infty \, |\lambda|^{-2d}, \quad \lambda \in [-\pi, \pi] \ . \tag{59}$$

By definition of $\Gamma^{(q,s)}$ in (55), we have, for all $\zeta \in \gamma_j\Gamma^{(q,s)}$, $\gamma_j^{-1}\sum_i \zeta_i - 2\pi s \in (-\pi, \pi]$. Hence using the (2π)-periodicity of $\widehat{h}_{j+m}$, we can use (4) for bounding $\widehat{h}_{j+m}(\gamma_j^{-1}\sum_i \zeta_i)$. With the change of variables $\zeta = \gamma_j\xi$ and (59), for all $\zeta \in \gamma_j\Gamma^{(q,s)}$ and j large enough so that $\gamma_{j+m}/\gamma_j \geq \overline{\gamma}_m/2$,

$$\gamma_j^{-(d(q)+K)} \, |f_{m,k}(\zeta; j, q)| = \gamma_j^{-(dq-q/2+1/2+K)} \, |f_{m,k}(\zeta; j, q)| \leq C_0 \, g(\zeta; 2\pi\gamma_j s) \ , \tag{60}$$

where C_0 is a positive constant and

$$g(\zeta; t) = \left(1 + \left|\sum_{i=1}^q \zeta_i - t\right|\right)^{-\alpha-K} \prod_{i=1}^q |\zeta_i|^{-d} \ .$$

The squared L^2-norm of $g(\cdot; t)$ reads

$$J(t) = \int_{\mathbb{R}^d} g^2(\zeta; t)\, \mathrm{d}\zeta = \int_{\mathbb{R}^q} \left(1 + \left|\sum_{i=1}^q \zeta_i - t\right|\right)^{-2\alpha-2K} \prod_{i=1}^q |\zeta_i|^{-2d} \prod_{i=1}^q \mathrm{d}\zeta_i \ .$$

We now show that Lemma 8.4 applies with $M_1 = 2\alpha + 2K$, $M_2 = 0$ and $\beta_i = 2d$ for $i = 1, \ldots, q$. Indeed, we have $M_2 - M_1 = -2\alpha - 2K \leq -2\alpha < -1$. Further, for all $\ell = 1, \ldots, q-1$, we have, by the assumption on d,

$$\sum_{i=\ell}^{q} \beta_i = 2d(1+q-\ell) > (1+q-\ell)(1-1/q) = q - \ell + (\ell-1)/q \geq q - \ell \,.$$

Finally, since $\alpha > 1/2$, one has $M_2 - M_1 + q = -2\alpha - 2K + q < q - 1 \leq \sum_i \beta_i$.

Applying Lemma 8.4, we get $J(t) \to 0$ as $|t| \to \infty$ and $J(0) < \infty$. Thus, if $s \neq 0$, one has $t = 2\pi\gamma_j s \to \infty$ as $j \to \infty$ and hence we obtain (58). If $s = 0$, then $t = 2\pi\gamma_j s = 0$ and using the bound (60), $J(0) < \infty$, and the dominated convergence theorem, we have that the convergence (57) follows from the convergence at a.e. $\zeta \in \mathbb{R}^q$ of the left hand side of (60), which we now establish. Recall that $f_{m,k}$ is defined in (54). By (6), (1) and the continuity of f^* at the origin, we have, as $j \to \infty$,

$$\begin{aligned}
\gamma_j^{-1/2} \widehat{h}_{j+m} \circ \Sigma_q(\zeta/\gamma_j) \quad &= \quad \left(\frac{\gamma_{j+m}}{\gamma_j}\right)^{1/2} \gamma_{j+m}^{-1/2}\, \widehat{h}_{j+m} \circ \Sigma_q\left((\zeta/\gamma_{j+m})(\gamma_{j+m}/\gamma_j)\right) \\
&\to \quad \bar{\gamma}_m^{1/2}\, \widehat{h}_\infty(\overline{\gamma}_m(\zeta_1 + \cdots + \zeta_q)) \,,
\end{aligned}$$

and for every $\ell = 1, \cdots, q$

$$\gamma_j^{-2d} f(\zeta_l/\gamma_j) = \gamma_j^{-2d} \left|1 - \mathrm{e}^{-\mathrm{i}\zeta_l/\gamma_j}\right|^{-2d} f^*(\zeta_l/\gamma_j) \to f^*(0)|\zeta_l|^{-2d} \,.$$

Hence $\gamma_j^{-(d(q)+K)} f_{m,k}(\zeta; j, q, 0) \mathbb{1}_{\Gamma(q,s)}(\gamma_j^{-1}\zeta)$ converges to

$$(\overline{\gamma}_m)^{1/2} (f^*(0))^{q/2}\, \frac{\mathrm{e}^{\mathrm{i}k\overline{\gamma}_m(\zeta_1+\cdots+\zeta_q)} \times \widehat{h}_\infty(\overline{\gamma}_m(\zeta_1 + \cdots + \zeta_q))}{(\mathrm{i}(\zeta_1 + \cdots + \zeta_q))^K |\zeta_1|^d \cdots |\zeta_q|^d} \,.$$

This concludes the proof. □

6.2. Proof of Proposition 6.2

We now consider the short-range dependence part of the wavelet coefficients $(W_{j,k})$ defined by (29) and (48). These wavelet coefficients can be equivalently defined as

$$W_{j,k}^{(SD)} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) \Delta^K Y_\ell^{(SD)} \,, \tag{61}$$

where we have set

$$\Delta^K Y_\ell^{(SD)} = \sum_{q \geq q_c + 1} \frac{c_q}{q!} H_q(X_\ell), \quad \ell \in \mathbb{Z} \,.$$

Using Lemma 4.1, since (26) holds and $\{H_q(X_\ell)\}_{\ell \in \mathbb{Z}}$ are uncorrelated weakly stationary processes, the process $\{\Delta^K Y_\ell^{(SD)}\}_{\ell \in \mathbb{Z}}$ is weakly stationary with spectral density

$$f^{(SD)}(\lambda) = \sum_{q \geq q_c + 1} \frac{c_q^2}{q!} f^{(\star q)}(\lambda), \quad \lambda \in (-\pi, \pi) \,.$$

By Lemma 4.1(ii), we have that $\|f^{(\star\{q_c+1\})}\|_\infty < \infty$. Using that $\|g_1 \star g_2\|_\infty \le \|g_1\|_\infty \|g_2\|_1$ and $\|f\|_1 = 1$ by assumption, an induction yields

$$\sup_{q>q_c} \|f^{(\star q)}\|_\infty \le \|f^{(\star\{q_c+1\})}\|_\infty \ .$$

Hence, by (26), we get $\|f^{(SD)}\|_\infty < \infty$. It follows that, for $W_{j,k}^{(SD)}$ defined in (61), there is a positive constant C such that,

$$\mathbb{E}[W_{j,k}^{(SD)2}] \le \|f^{(SD)}\|_\infty \int_{-\pi}^{\pi} |\widehat{h}_j^{(K)}(\lambda)|^2 \mathrm{d}\lambda \le C \int_0^{\pi} |\lambda|^{-2K} \ |\widehat{h}_j(\lambda)|^2 \mathrm{d}\lambda = O(\gamma_j^{2K}) \ ,$$

where we used (4) with $M \ge K$ and $\alpha > 1/2$. This last relation implies (51) and concludes the proof of Proposition 6.2. $\square$

7. Proof of Lemmas 5.1 and 5.2

7.1. Proof of Lemma 5.1

Let us first prove that if $\theta \in \mathcal{S}_{q,d}^{(K)}$ then $Z_{q,d}^{(K)}(\theta)$ exists. Indeed, by Definition 5.1, $Z_{q,d}^{(K)}(\theta)$ exists if

$$\int_{\mathbb{R}^q} \frac{|\widehat{\theta}(u_1+\cdots+u_q)|^2}{|u_1+\cdots+u_q|^{2K}|u_1\cdots u_q|^{2d}} \mathrm{d}u_1\cdots\mathrm{d}u_q < \infty \ . \tag{62}$$

Use now Lemma 8.3 with $\beta_1 = \cdots = \beta_q = -2d$ and $f(x) = |\widehat{\theta}(x)|^2/|x|^{2K}$ and deduce that Condition (62) is equivalent to

$$\Gamma \int_{\mathbb{R}} |\widehat{\theta}(s)|^2 |s|^{q-1-2qd-2K} \mathrm{d}s < \infty \ , \tag{63}$$

where

$$\Gamma = \prod_{i=2}^{q} \left(\int_{\mathbb{R}} |t|^{q-i-2d(q-i+1)} |1-t|^{-2d} \mathrm{d}t \right) \ .$$

Note that the conditions $0 < d < 1/2$ and $0 < q < 1/(1-2d)$ ensure that Γ is finite. Further, Relation (63) implies $\theta \in \mathcal{S}_{q,d}^{(K)}$.

We now prove that for any m, k, $h_{\infty,m,k} \in \mathcal{S}_{q,d}^{(K)}$ when $K \in \{0,\ldots,M\}$. By Definition (40) of $h_{\infty,m,k}$

$$\widehat{h}_{\infty,m,k}(\xi) = (\overline{\gamma}_m)^{1/2} \ \mathrm{e}^{-\mathrm{i}\overline{\gamma}_m \xi} \ \overline{\widehat{h}_\infty(\overline{\gamma}_m \xi)} \ .$$

Hence

$$\int_{\mathbb{R}} |\widehat{h}_{\infty,m,k}(s)|^2 |s|^{q-1-2qd-2K} \mathrm{d}s = \overline{\gamma}_m \int_{\mathbb{R}} |\widehat{h}_\infty(\overline{\gamma}_m s)|^2 |s|^{q-1-2qd-2K} \mathrm{d}s \ .$$

Set $v = \overline{\gamma}_m s$ and deduce that $h_{\infty,m,k} \in \mathcal{S}_{q,d}^{(K)}$ is equivalent to

$$\overline{\gamma}_m^{2-(q-1-2qd-2K)} \int_{\mathbb{R}} |\widehat{h}_\infty(v)|^2 |v|^{q-1-2qd-2K} \mathrm{d}v < \infty \ .$$

Assumption (12) implies that

$$\int_{\mathbb{R}} |\widehat{h}_\infty(v)|^2 |v|^{q-1-2qd-2K} \mathrm{d}v \leq \int_{\mathbb{R}} \frac{|v|^{2M}}{(1+|v|)^{2M+2\alpha}} |v|^{q-1-2qd-2K} \mathrm{d}v \ .$$

Since $M \geq K$ and $q(1-2d) \in (0,1)$ then $2M+q-1-2qd-2K = (2M-2K)+q(1-2d)-1 > -1$. Further $\alpha > 1/2$ and $q(1-2d) \in (0,1)$ imply that $2M-2M-2\alpha+(q-1-2qd-2K) = -2\alpha-2K+q(1-2d)-1 < -1$. Then

$$\int_{\mathbb{R}} |\widehat{h}_{\infty,m,k}(s)|^2 |s|^{q-1-2qd-2K} \mathrm{d}s < \infty \ .$$

holds and $h_{\infty,m,k} \in \mathcal{S}^{(K)}_{q,d}$.

7.2. Proof of Lemma 5.2

Let $a_t(u_1,\cdots,u_q)$ denote the kernel of the integral in (42) defining $\tilde{Z}^{(K)}_{q,d}$ and suppose we can exchange the order of integration and write

$$\int_{\mathbb{R}} \tilde{Z}^{(K)}_{q,d}(t)\theta(t)\mathrm{d}t = \int_{\mathbb{R}^q}^{\prime\prime} \left[\int_{\mathbb{R}} a_t(u_1,\cdots,u_q)\theta(t)\mathrm{d}t\right] \mathrm{d}\widehat{W}(u_1)\cdots\mathrm{d}\widehat{W}(u_q) \ . \tag{64}$$

Then condition (43) gives

$$\int_{\mathbb{R}} \left[\mathrm{e}^{\mathrm{i}t\,(u_1+\cdots+u_q)} - \sum_{\ell=0}^{K-1} \frac{(\mathrm{i}t(u_1+\cdots+u_q))^\ell}{\ell!}\right] \overline{\theta(t)}\,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\,(u_1+\cdots+u_q)}\,\overline{\theta(t)}\,\mathrm{d}t = \overline{\widehat{\theta}\circ\Sigma_q(u)} \ ,$$

showing that (64) equals $\tilde{Z}^{(K)}_{q,d}(\theta)$ defined in (39). It remains to justify the change of order of integration in (64) by using a stochastic Fubini theorem, (see for instance [20, Theorem 2.1]). A sufficient condition is

$$\int_{\mathbb{R}} \left(a_t^2(u_1,\cdots,u_q)\mathrm{d}u_1\cdots\mathrm{d}u_q\right)^{1/2} \mathrm{d}t < \infty \ .$$

This condition is satisfied, because setting $v = tu$, we have

$$\int_{\mathbb{R}^q} \left|\mathrm{e}^{\mathrm{i}t\,(u_1+\cdots+u_q)} - \sum_{\ell=0}^{K-1} \frac{(\mathrm{i}t(u_1+\cdots+u_q))^\ell}{\ell!}\right|^2 |\mathrm{i}(u_1+\cdots+u_q)|^{-2K} |u_1\cdots u_q|^{-2d}\,\mathrm{d}^q u \ ,$$

$$\leq |t|^{2K+2d-q} \int_{\mathbb{R}^q} (1+|u_1+\cdots+u_q|)^{-2K} |u_1\cdots u_q|^{-2d} \mathrm{d}^q u \ .$$

8. Auxiliary lemmas

The following lemma provides a bound for the convolution of two functions exploding at the origin and decaying polynomially at infinity.

Lemma 8.1. *Let $\alpha > 1$ and $\beta_1, \beta_2 \in [0, 1)$ such that $\beta_1 + \beta_2 < 1$, and set*

$$g_i(t) = |t|^{-\beta_i}(1 + |t|)^{\beta_i - \alpha} .$$

Then

$$\sup_{u \in \mathbb{R}} \left((1 + |u|)^{\alpha} \int_{\mathbb{R}} g_1(u - t) g_2(t) \, \mathrm{d}t \right) < \infty . \tag{65}$$

Proof. We first show that

$$J(u) = \int_{\mathbb{R}} g_1(u - t) g_2(t) \, \mathrm{d}t = \int_{\mathbb{R}} |u - t|^{-\beta_1}(1 + |u - t|)^{\beta_1 - \alpha} |t|^{-\beta_2}(1 + |t|)^{\beta_2 - \alpha} \, \mathrm{d}t$$

is uniformly bounded on $\mathbb{R}$. Using the assumptions on β_1, β_2, there exist $p > 1$ such that $\beta_1 < 1/p < 1 - \beta_2$. Let q be such that $1/p + 1/q = 1$. The Hölder inequality implies that

$$J(u)^{pq} \le \int_{\mathbb{R}} |t|^{-p\beta_1}(1 + |t|)^{p\beta_1 - p\alpha} \, \mathrm{d}t \times \int_{\mathbb{R}} |t|^{-q\beta_2}(1 + |t|)^{q\beta_2 - q\alpha} \, \mathrm{d}t .$$

The condition on $\alpha, \beta_1, \beta_2, p$ and the definition of q imply that these two integrals are finite. Hence $\sup_u J(u) < \infty$.

We now determine how fast $J(u)$ tends to 0 as $u \to \infty$. Observe that, if $|t - u| \le |u|/2$, then $|t| \ge |u|/2$. By splitting the integral in two integrals on the domains $|t - u| \le |u|/2$ and $|t - u| > |u|/2$, we get $J(u) \le J_1(u) + J_2(u)$ with

$$J_1(u) \le (|u|/2)^{-\beta_2}(1 + |u|/2)^{\beta_2 - \alpha} \int_{\mathbb{R}} |u - t|^{-\beta_1}(1 + |t - u|)^{\beta_1 - \alpha} \mathrm{d}t ,$$

and

$$J_2(u) \le (|u|/2)^{-\beta_1}(1 + |u|/2)^{\beta_1 - \alpha} \int_{\mathbb{R}} |t|^{-\beta_2}(1 + |t|)^{\beta_2 - \alpha} \mathrm{d}t .$$

Now, as $|u| \to \infty$, we have $J_i(u) = O(|u|^{-\alpha})$ for $i = 1, 2$, which achieves the proof. □

The next lemma describes the convolutions of two periodic functions that explode at the origin as a power. A different definition of convolution is involved here (see (33)).

Lemma 8.2. *Let $(\beta_1, \beta_2) \in (0, 1)^2$. Let g_1, g_2 be (2π)-periodic functions such that $g_i(\lambda) = |\lambda|^{-\beta_i} \, g_i^*(\lambda)$, $i = 1, 2$. Each $g_i^*(\lambda)$ is a (2π)-periodic non-negative function, bounded on $(-\pi, \pi)$ and positive at the origin, where it is also continuous. Let $g = g_1 \star g_2$ as defined in (33). Then,*

- *If $\beta_1 + \beta_2 < 1$, g is bounded and continuous on $(-\pi, \pi)$, and satisfies $g(0) > 0$.*

- *If* $\beta_1+\beta_2>1$,
$$g(\lambda)=|\lambda|^{-(\beta_1+\beta_2-1)}g^*(\lambda)\ ,$$
where $g^*(\lambda)$ *is bounded on* $(-\pi,\pi)$ *and converges to a positive constant as* $\lambda\to 0$. *If moreover for some* $\beta\in(0,2]$ *such that* $\beta<\beta_1+\beta_2-1$ *and some* $L>0$, *one has for any* $i\in\{1,2\}$
$$|g_i^*(\lambda)-g_i^*(0)|\leq L|\lambda|^\beta,\ \forall\lambda\in(-\pi,\pi)\ , \tag{66}$$
then there exists some $L'>0$ *depending only on* L,β_1,β_2 *such that*
$$|g^*(\lambda)-g^*(0)|\leq L'|\lambda|^\beta,\ \forall\lambda\in(-\pi,\pi)\ .$$

Proof. By (33) and (2π)-periodicity, we may write
$$g(\lambda)=\int_{-\pi}^{\pi}g_1(u)g_2(\lambda-u)\ \mathrm{d}u=\int_{-\pi}^{\pi}|\{\lambda-u\}|^{-\beta_1}g_1^*(\lambda-u)\ |u|^{-\beta_2}g_2^*(u)\ \mathrm{d}u\ . \tag{67}$$

Let us first consider the case $\beta_1+\beta_2<1$. We clearly have $g(0)>0$. To prove that g is bounded, we proceed as in the case of convolutions of non-periodic functions (see the proof of Lemma 8.1), namely, for p,q such that $\beta_1<1/p<1-\beta_2$ and $1/p+1/q=1$, the Hölder inequality gives that
$$\|g\|_\infty^{pq}\leq\|g_1\|_p^p\ \|g_2\|_q^q\leq\|g_1^*\|_\infty^p\ \|g_2^*\|_\infty^q\int_{-\pi}^{\pi}|t|^{-p\beta_1}\,\mathrm{d}t\times\int_{-\pi}^{\pi}|t|^{-q\beta_2}\,\mathrm{d}t<\infty\ . \tag{68}$$
For any $\epsilon>0$ and $i=1,2$, let $g_{\epsilon,i}$ be the (2π)-periodic function such that for all $\lambda\in(-\pi,\pi)$, $g_{\epsilon,i}(\lambda)=\mathbb{1}_{(-\epsilon,\epsilon)}(\lambda)\ g_i(\lambda)$ and let $\bar{g}_{\epsilon,i}=g_i-g_{\epsilon,i}$. Then $g=\bar{g}_{\epsilon,1}\star\bar{g}_{\epsilon,2}+g_{\epsilon,1}\star\bar{g}_{\epsilon,2}+\bar{g}_{\epsilon,1}\star g_{\epsilon,2}+g_{\epsilon,1}\star g_{\epsilon,2}$. Since $\bar{g}_{\epsilon,i}$ is bounded for $i=1,2$, we have that $\bar{g}_{\epsilon,1}\star\bar{g}_{\epsilon,2}$ is continuous. On the other hand, using the Hölder inequality as in (68), we get that $\|g_{\epsilon,1}\star\bar{g}_{\epsilon,2}\|_\infty$, $\|\bar{g}_{\epsilon,1}\star g_{\epsilon,2}\|_\infty$, $\|\bar{g}_{\epsilon,1}\star\bar{g}_{\epsilon,2}\|_\infty$ tend to zero as $\epsilon\to 0$. Hence g is continuous as well.

We now consider the case $\beta_1+\beta_2\geq 1$. Setting $v=u/\lambda$ in (67), we get, for any $\lambda\in[-\pi,\pi]\backslash\{0\}$,
$$g^*(\lambda)=|\lambda|^{\beta_1+\beta_2-1}g(\lambda)=\int_{\mathbb{R}}\mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v)\ |\{(1-v)\}_\lambda|^{-\beta_1}|v|^{-\beta_2}g_1^*(\lambda(1-v))g_2^*(\lambda v)\ \mathrm{d}v\ ,$$
where for any real number x and $\lambda\neq 0$, $\{x\}_\lambda$ denotes the unique element of $[-\pi/|\lambda|,\pi/|\lambda|]$ such that $x-\{x\}_\lambda\in\mathbb{Z}$. Take now $|\lambda|$ small enough so that $\pi/|\lambda|>2$. Then, for any $v\in(-\pi/|\lambda|+1,\pi/|\lambda|]$, we have $|\{(1-v)\}_\lambda|=|1-v|\geq|1-|v||$ and, for any $v\in(-\pi/|\lambda|,-\pi/|\lambda|+1]$, we have
$$|\{(1-v)\}_\lambda|=|1-v-2\pi/|\lambda||=2\pi/|\lambda|+v-1\geq -v-1=|1-|v||\ . \tag{69}$$
Thus we have $\mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v)\ |\{(1-v)\}_\lambda|^{-\beta_1}|\leq|1-|v||^{-\beta_1}$ for all $v\in\mathbb{R}$. We conclude that for $|\lambda|$ small enough, the integrand in the last display is bounded from above by

$|1-|v||^{-\beta_1}|v|^{-\beta_2}\ \|g_1^*\|_\infty\|g_2^*\|_\infty$, which is integrable on $v\in\mathbb{R}$. Hence g^* is bounded, and by dominated convergence, as $\lambda\to 0$,

$$g^*(\lambda)\to g_1^*(0)g_2^*(0)\int_{\mathbb{R}}|1-v|^{-\beta_1}\ |v|^{-\beta_2}\ \mathrm{d}v>0\ . \tag{70}$$

We set $g^*(0)$ equal to this limit.

Suppose moreover that g_1^*, g_2^* satisfy (66). We take $g_1^*(0)=g_2^*(0)=1$ without loss of generality and denote $r_i(\lambda)=|g_i^*(\lambda)-1|$ for $i=1,2$. Then $r(\lambda)=|g^*(\lambda)-g^*(0)|$, where $g^*(0)$ is defined as the limit in (70) , is at most

$$\int_{\mathbb{R}}\left|\mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v)\ |\{(1-v)\}_\lambda|^{-\beta_1}|v|^{-\beta_2}g_1^*(\lambda(1-v))g_2^*(\lambda v)-|1-v|^{-\beta_1}\ |v|^{-\beta_2}\right|\ \mathrm{d}v\ .$$

Setting $g_i^*(\lambda)=(g_i^*(\lambda)-1)+1$, we have $r\le A+B_1+B_2+C$ with

$$A(\lambda)=\int_{\mathbb{R}}\left|\mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v)\ |\{(1-v)\}_\lambda|^{-\beta_1}|v|^{-\beta_2}-|1-v|^{-\beta_1}\ |v|^{-\beta_2}\right|\ \mathrm{d}v\ ,$$
$$B_i(\lambda)=\int_{\mathbb{R}}\mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v)\ |\{(1-v)\}_\lambda|^{-\beta_j}|v|^{-\beta_i}r_i(\lambda v)\ \mathrm{d}v\ ,$$

where (i,j) is $(1,2)$ or $(2,1)$, and

$$C(\lambda)=\int_{\mathbb{R}}\mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v)\ |\{(1-v)\}_\lambda|^{-\beta_1}|v|^{-\beta_2}r_1(\lambda(1-v))r_2(\lambda v)\ \mathrm{d}v\ .$$

Since $\{(1-v)\}_\lambda=1-v$ for $v\in[-\pi/|\lambda|+1,\pi/|\lambda|)$ and λ large enough, we have

$$A(\lambda)=\int_{(-\pi/|\lambda|,\pi/|\lambda|)^c}|1-v|^{-\beta_1}\ |v|^{-\beta_2}\ \mathrm{d}v \\ +\int_{-\pi/|\lambda|}^{-\pi/|\lambda|+1}\left||\{(1-v)\}_\lambda|^{-\beta_1}|v|^{-\beta_2}-|1-v|^{-\beta_1}\ |v|^{-\beta_2}\right|\ \mathrm{d}v\ .$$

The first integral is $O(|\lambda|^{\beta_1+\beta_2-1})$. Using (69), the second line of the last display is less than

$$\int_{\pi/|\lambda|-1}^{\pi/|\lambda|}\left[|1-v|^{-\beta_1}|v|^{-\beta_2}+|1+v|^{-\beta_1}\ v^{-\beta_2}\right]\ \mathrm{d}v=O(|\lambda|^{\beta_1+\beta_2})\ .$$

We conclude that as $\lambda\to 0$, $A(\lambda)=O(|\lambda|^{\beta_1+\beta_2-1})$. Moreover using that $r_i(\lambda)\le L|\lambda|^\beta$ and $\beta_1+\beta_2-\beta>1$, we have $B_i(\lambda)=O(|\lambda|^\beta)$ for $i=1,2$. The same is true for C since r_1 and r_2 are also bounded on $\mathbb{R}$. This achieves the proof. □

Lemma 8.3. *Let p be a positive integer and $f:\mathbb{R}\to\mathbb{R}_+$. Then, for any $\beta\in\mathbb{R}^q$,*

$$\int_{\mathbb{R}^q}f(y_1+\cdots+y_q)\prod_{i=1}^q|y_i|^{\beta_i}\ \mathrm{d}y_1\cdots\mathrm{d}y_q=\Gamma\times\int_{\mathbb{R}}f(s)|s|^{q-1+\beta_1+\cdots+\beta_q}\mathrm{d}s\ , \tag{71}$$

where, for all $i \in \{1, \cdots, q\}$, $B_i = \beta_i + \cdots + \beta_q$ and

$$\Gamma = \prod_{i=2}^{q} \left(\int_{\mathbb{R}} |t|^{q-i+B_i} |1-t|^{\beta_{i-1}} \mathrm{d}t \right) .$$

(We note that Γ may be infinite in which case (71) holds with the convention $\infty \times 0 = 0$).

Proof. Relation (71) is obtained by using the following two successive change of variables followed by an application of the Fubini Theorem. Setting, for all $i = 1, \cdots, q$, $u_i = \sum_{j=i}^{q} y_j$, we get that $y_i = u_i - u_{i+1}$ for $i < q$ and $y_q = u_q$. Then the integral in the left–hand side of (71) reads

$$\int_{\mathbb{R}^q} f(u_1) \left[|u_q|^{\beta_q} \prod_{i=1}^{q-1} |u_i - u_{i+1}|^{\beta_i} \right] \mathrm{d}u_1 \cdots \mathrm{d}u_q \,. \tag{72}$$

The second change of variables consists in setting, for all $i = 1, \cdots, q$, $u_i = \prod_{j=1}^{i} t_j$. Then

$$\mathrm{d}u_1 \cdots \mathrm{d}u_q = \left(\prod_{i=1}^{q-1} t_i^{q-i} \right) \mathrm{d}t_1 \cdots \mathrm{d}t_q,$$

$$\prod_{i=1}^{q-1} |u_i - u_{i+1}|^{\beta_i} = \prod_{i=1}^{q-1} \left(|t_1 \cdots t_i|^{\beta_i} |1 - t_{i+1}|^{\beta_i} \right) = \left(\prod_{i=1}^{q-1} |t_i|^{\beta_i + \cdots + \beta_{q-1}} \right) \left(\prod_{i=2}^{q} |1 - t_i|^{\beta_{i-1}} \right),$$

and $|u_q| = \prod_{i=1}^{q} |t_i|^{\beta_q}$, so that (72) becomes

$$\int_{\mathbb{R}^q} f(t_1) \prod_{i=1}^{q} |t_i|^{\beta_i + \cdots + \beta_q + q - i} \prod_{i=2}^{q} |1 - t_i|^{\beta_{i-1}} \mathrm{d}t_1 \cdots \mathrm{d}t_q \,,$$

which by Fubini Theorem yields the required result.

□

Lemma 8.4. *Let $a \in \mathbb{R}$ and q be a positive integer. Let $\beta = (\beta_1, \cdots, \beta_q) \in (-\infty, 1)^q$, $M_1 > 0$ and $M_2 > -1$ such that $M_2 - M_1 < -1$. Assume that $q + M_2 - M_1 < \sum_{i=1}^{q} \beta_i$, and that for any $\ell \in \{1, \cdots, q-1\}$, $\sum_{i=\ell}^{q} \beta_i > q - \ell$. Set for any $a \in \mathbb{R}$,*

$$J_q(a; M_1, M_2; \beta) = \int_{\mathbb{R}^q} \frac{|\Sigma_q(\zeta) - a|^{M_2}}{(1 + |\Sigma_q(\zeta) - a|)^{M_1} \prod_{i=1}^{q} |\zeta_i|^{\beta_i}} \, \mathrm{d}\zeta.$$

Then one has

$$\sup_{a \in \mathbb{R}} (1 + |a|)^{1 - q + \sum_{i=1}^{q} \beta_i} J_q(a; M_1, M_2; \beta) < \infty \,. \tag{73}$$

In particular,

$$J_q(0; M_1, M_2; \beta) < \infty,$$

and

$$J_q(a; M_1, M_2; \beta) = O(|a|^{-(1-q+\sum_{i=1}^q \beta_i)}) \quad \textit{as } a \to \infty\ .$$

Proof. Since $J_q(a; M_1, M_2; \beta_1, \cdots, \beta_q) = J_q(-a; M_1, M_2; \beta)$, we may suppose $a \geq 0$. By Lemma 8.3,

$$J_q(a; M_1, M_2; \beta_1, \cdots, \beta_q) = \Gamma \int_{\mathbb{R}} \frac{|s-a|^{M_2}|s|^{q-1-(\beta_1+\cdots+\beta_q)}}{(1+|s-a|)^{M_1}} \mathrm{d}s$$

where

$$\Gamma = \prod_{i=2}^q \int_{\mathbb{R}} \frac{\mathrm{d}t}{|t|^{\beta_i+\cdots+\beta_q-(q-i)}|1-t|^{\beta_{i-1}}}\ .$$

The conditions on β_i's, M_1 and M_2 imply $J_q(a; M_1, M_2; \beta_1, \cdots, \beta_q) < \infty$ for all a. To obtain the sup on $a > 0$, we set $v = s/a$. Then, denoting $S = \sum_{i=1}^q \beta_i$, we get

$$J_q(a; M_1, M_2; \beta) = Ca^{q+M_2-S} \int_{\mathbb{R}} |v-1|^{M_2}(1+a|v-1|)^{-M_1}|v|^{-S+(q-1)}\mathrm{d}v\ , \tag{74}$$

where C is a positive constant. We separate the integration domain in two. Suppose first that $|v-1| \leq a^{-1}$. Then in this case we have $(1+a|v-1|)^{-M_1} \leq 1$. Since $|v|$ is bounded on the interval $|v-1| < a^{-1}$ for a large then as $a \to \infty$,

$$\int_{|v-1|\leq a^{-1}} |v-1|^{M_2}(1+a|v-1|)^{-M_1}|v|^{-S+(q-1)}\mathrm{d}v = O\left(\int_{|v-1|\leq a^{-1}} |v-1|^{M_2}\mathrm{d}v\right) = O(a^{-1-M_2})\ .$$

Now suppose that $|v-1 > a^{-1}|$. Then $(1+a|v-1|)^{-M_1} \leq (a|v-1|)^{-M_1}$, and

$$\begin{aligned} I &= \int_{|v-1|>a^{-1}} |v-1|^{M_2}(1+a|v-1|)^{-M_1}|v|^{-S+(q-1)}\mathrm{d}v \\ &\leq a^{-M_1}\int_{|v-1|>a^{-1}} |v-1|^{M_2-M_1}|v|^{-S+(q-1)}\mathrm{d}v \\ &= a^{-M_1}\left(\int_{|v|\geq 2} |v-1|^{M_2-M_1}|v|^{-S+(q-1)}\mathrm{d}v + \int_{1/2\leq|v|\leq 2, |v-1|>a^{-1}} |v-1|^{M_2-M_1}|v|^{-S+(q-1)}\mathrm{d}v\right) \\ &\quad +a^{-M_1}\int_{|v|\leq 1/2, |v-1|>a^{-1}} |v-1|^{M_2-M_1}|v|^{-S+(q-1)}\mathrm{d}v\ . \end{aligned}$$

The first integral concentrates around $v = \infty$, the second around $v = 1$ and the third around $v = 0$. The first integral is bounded, the second is

$$O(\int_{|v-1|>a^{-1}} |v-1|^{M_2-M_1}\mathrm{d}v) = O(a^{M_1-M_2-1}), \quad \text{as } a \to \infty,$$

and the third is bounded. Therefore we get

$$I = O(a^{-M_1}) + O(a^{-M_2-1}),$$

since $M_2 - M_1 < -1$. Thus (74) gives

$$J_q(a; M_1, M_2; \beta) = O(a^{-1+q-S}) \quad \text{as } a \to \infty,$$

yielding the bound (73). $\square$

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[1] E. Moulines, F. Roueff, M. S. Taqqu, On the spectral density of the wavelet coefficients of long memory time series with application to the log-regression estimation of the memory parameter, J. Time Ser. Anal. 28 (2) (2007) 155–187.

[2] F. Roueff, M. S. Taqqu, Central limit theorems for arrays of decimated linear processes, Stoch. Proc. App. 119 (9) (2009) 3006–3041.

[3] Y. Meyer, Ondelettes et opérateurs, Hermann, 1990.

[4] S. Mallat, A wavelet tour of signal processing, Academic Press Inc., San Diego, CA, 1998.

[5] G. W. Wornell, A. V. Oppenheim, Estimation of fractal signals from noisy measurements using wavelets, IEEE Trans. Signal Process. 40 (3) (1992) 611 – 623.

[6] P. Flandrin, On the spectrum of fractional Brownian motions, IEEE Transactions on Information Theory IT-35 (1) (1989) 197–199.

[7] P. Flandrin, Some aspects of nonstationary signal processing with emphasis on time-frequency and time-scale methods, in: J. Combes, A. Grossman, P. Tchamitchian (Eds.), Wavelets, Springer-Verlag, 1989, pp. 68–98.

[8] P. Flandrin, Fractional Brownian motion and wavelets, in: M. Farge, J. Hung, J. Vassilicos (Eds.), Fractals and Fourier Transforms-New Developments and New Applications, Oxford University Press, 1991.

[9] P. Flandrin, Time-Frequency/Time-scale Analysis, 1st Edition, Academic Press, 1999.

[10] P. Abry, D. Veitch, Wavelet analysis of long-range-dependent traffic, IEEE Trans. Inform. Theory 44 (1) (1998) 2–15.

[11] P. Abry, D. Veitch, P. Flandrin, Long-range dependence: revisiting aggregation with wavelets, J. Time Ser. Anal. 19 (3) (1998) 253–266.

[12] R. Fox, M. S. Taqqu, Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series, Ann. Statist. 14 (2) (1986) 517–532.

[13] P. M. Robinson, Log-periodogram regression of time series with long range dependence, The Annals of Statistics 23 (1995) 1048–1072.

[14] P. M. Robinson, Gaussian semiparametric estimation of long range dependence, Ann. Statist. 23 (1995) 1630–1661.

[15] G. Faÿ, E. Moulines, F. Roueff, M. Taqqu, Estimators of long-memory: Fourier versus wavelets, J. of Econometrics (In press, DOI:10.1016/j.jeconom.2009.03.005).

[16] J. Bardet, C. Tudor, A wavelet analysis of the rosenblatt process: chaos expansion and estimation of the self-similarity parameter, Submitted.

[17] P. Major, Multiple Wiener-Itô integrals, Vol. 849 of Lecture Notes in Mathematics, Springer, Berlin, 1981.

[18] D. Nualart, The Malliavin Calculus and Related Topics, Springer, 2006.

[19] M. Taqqu, Central limit theorems and other limit theorems for functionals of gaussian processes., Z. Wahrsch. verw. Gebiete 70 (1979) 191–212.

[20] V. Pipiras, M. S. Taqqu, Regularization and integral representations of Hermite processes, in preparation.