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► **To cite this version:**

Marianne Clausel, François Roueff, Murad Taqqu, Ciprian Tudor. Large scale behavior of wavelet coefficients of non-linear subordinated processes with long memory. Applied and Computational Harmonic Analysis, Elsevier, 2012, 32 (2), pp.223-241. 10.1016/j.acha.2011.04.003 . hal-00491303

**HAL Id: hal-00491303**

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Submitted on 11 Jun 2010

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# Large scale behavior of wavelet coefficients of non-linear subordinated processes with long memory

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## Abstract

We study the asymptotic behavior of wavelet coefficients of random processes with long memory. These processes may be stationary or not and are obtained as the output of non-linear filter with Gaussian input. The wavelet coefficients that appear in the limit are random, typically non-Gaussian and belong to a Wiener chaos. They can be interpreted as wavelet coefficients of a generalized self-similar process.

*Keywords:* Hermite processes, Wavelet coefficients, Wiener chaos, self-similar processes, Long-range dependence.

*2000 MSC:* Primary 42C40, 60G18, 62M15, Secondary: 60G20, 60G22

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## 1. Introduction

Let  $X = \{X_n\}_{n \in \mathbb{Z}}$  be a stationary Gaussian process with mean zero, unit variance and spectral density  $f(\lambda)$ ,  $\lambda \in (-\pi, \pi]$  and thus covariance equal to

$$r(n) = \mathbb{E}(X_0 X_n) = \int_{-\pi}^{\pi} e^{in\lambda} f(\lambda) d\lambda .$$

The process  $\{X_n\}_{n \in \mathbb{Z}}$  is said to have *short memory* or *short-range dependence* if  $f(\lambda)$  is bounded around  $\lambda = 0$  and *long memory* or *long-range dependence* if  $f(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow 0$ . We will suppose that  $\{X_n\}_{n \in \mathbb{Z}}$  has long-memory with memory parameter  $d > 0$ , that is,

$$f(\lambda) \sim |\lambda|^{-2d} f^*(\lambda) \text{ as } \lambda \rightarrow 0$$

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where  $f^*(\lambda)$  is a bounded spectral density which is continuous and positive at the origin. It is convenient to interpret this behavior as the result of a fractional integrating operation, whose transfer function reads  $\lambda \mapsto (1 - e^{-i\lambda})^{-d}$ . Hence we set

$$f(\lambda) = |1 - e^{-i\lambda}|^{-2d} f^*(\lambda), \quad \lambda \in (-\pi, \pi]. \quad (1)$$

We relax the above assumptions in two ways :

1. Consider, instead of the Gaussian process  $\{X_n\}_{n \in \mathbb{Z}}$  the non-Gaussian process  $\{G(X_n)\}_{n \in \mathbb{Z}}$  where  $G$  is a non-linear filter such that  $\mathbb{E}[G(X_n)] = 0$  and  $\mathbb{E}[G(X_n)^2] < \infty$ . The non-linear process  $\{G(X_n)\}_{n \in \mathbb{Z}}$  is said to be subordinated to the Gaussian process  $\{X_n\}_{n \in \mathbb{Z}}$ .
2. Drop the stationarity assumption by considering a process  $\{Y_n\}_{n \in \mathbb{Z}}$  which becomes stationary when differenced  $K \geq 0$  times.

We shall thus consider  $\{Y_n\}_{n \in \mathbb{Z}}$  such that

$$(\Delta^K Y)_n = G(X_n), \quad n \in \mathbb{Z},$$

where  $(\Delta Y)_n = Y_n - Y_{n-1}$  and where  $\{X_n\}_{n \in \mathbb{Z}}$  is Gaussian with spectral density  $f$  satisfying (1).

Since  $Y = \{Y_n\}_{n \in \mathbb{Z}}$  is random so will be its wavelet coefficients  $\{W_{j,k}, j \geq 0, k \in \mathbb{Z}\}$  which are defined below. Our goal is to find the distribution of the wavelet coefficients at large scales  $j \rightarrow \infty$ . This is an important step in developing methods for estimating the underlying long memory parameter  $d$ . The large scale behavior of the wavelet coefficients was studied in [1] in the case where there was no filter  $G$ , that is, when  $Y$  is a Gaussian process such that  $\Delta^K Y = X$ , and also in the case where  $Y$  is a non-Gaussian linear process (see [2]).

We obtain our random wavelet coefficients by using more general linear filters that those related to multiresolution analysis (MRA) (see for e.g. [3], [4]). In practice, however, the methods are best implemented using Mallat's algorithm and a MRA. Our filters are denoted  $h_j$  where  $j$  is the scale and we use a scaling factor  $\gamma_j \uparrow \infty$  as  $j \uparrow \infty$ . In the case of a MRA,  $\gamma_j = 2^j$  and  $h_j$  are generated by a (low pass) scaling filter and its corresponding quadratic (high pass) mirror filter. More generally one can use a scaling function  $\varphi$  and a mother wavelet  $\psi$  to generate the random wavelet coefficients by setting

$$W_{j,k} = \int_{\mathbb{R}} \psi_{j,k}(t) \left( \sum_{\ell \in \mathbb{Z}} \varphi(t - \ell) Y_\ell \right) dt, \quad (2)$$

where  $\psi_{j,k} = 2^{-j/2} \psi(2^{-j}t - k)$ ,  $j \geq 0$ . Observe that we use here the engineering convention that large values of  $j$  correspond to large scales and hence low frequencies. If  $\varphi$  and  $\psi$  have compact support then the corresponding filters  $h_j$  have finite support of size  $O(2^j)$ . For more details on related conditions on  $\varphi$  and  $\psi$  (see [1]).

The idea of using wavelets to estimate the long memory coefficient  $d$  goes back to Wornell and al. ([5]) and Flandrin ([6, 7, 8, 9]). See also Abry and al. ([10, 11]). Those methods are an alternative to the Fourier methods developed by Fox and Taqqu ([12]) and Robinson ([13, 14]). For a general comparison of Fourier and wavelet approach, see [15]. The case of the Rosenblatt process, which is the Hermite process of order  $q = 2$ , was studied by [16].

The paper is structured as follows. In Section 2, we introduce the wavelet filters. The processes are defined in Section 3 using integral representations and Section 4 presents the so-called Wiener chaos decomposition. The main result and its interpretations is given in Section 5. It is proved in Section 6. Auxiliary lemmas are presented and proved in Sections 7 and 8.

## 2. Assumptions on the wavelet filter

The wavelet transform of  $Y$  involves the application of a linear filter  $h_j(\tau), \tau \in \mathbb{Z}$ , at each scale  $j \geq 0$ . We shall characterize the filters  $h_j$  by their discrete Fourier transform :

$$\widehat{h}_j(\lambda) = \sum_{\tau \in \mathbb{Z}} h_j(\tau) e^{-i\lambda\tau}, \quad \lambda \in [-\pi, \pi].$$

Assumptions on  $\widehat{h}_j$  are stated below. The resulting wavelet coefficients are defined as

$$W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j(\gamma_j k - \ell) Y_\ell, \quad j \geq 0, \quad k \in \mathbb{Z},$$

where  $\gamma_j \uparrow \infty$  is a sequence of non-negative scale factors applied at scale  $j$ , for example  $\gamma_j = 2^j$ . We will assume that for any  $m \in \mathbb{Z}$ ,

$$\lim_{j \rightarrow \infty} \frac{\gamma_{j+m}}{\gamma_j} = \bar{\gamma}_m > 0. \quad (3)$$

As noted, in this paper, we do not assume that the wavelet coefficients are orthogonal nor that they are generated by a multiresolution analysis. Our assumptions on the filters  $h_j$  are as follows :

- a. Finite support: For each  $j$ ,  $\{h_j(\tau)\}_{\tau \in \mathbb{Z}}$  has finite support.
- b. Uniform smoothness: There exists  $M \geq K$ ,  $\alpha > 1/2$  and  $C > 0$  such that for all  $j \geq 0$  and  $\lambda \in [-\pi, \pi]$ ,

$$|\widehat{h}_j(\lambda)| \leq \frac{C \gamma_j^{1/2} |\gamma_j \lambda|^M}{(1 + \gamma_j |\lambda|)^{M+\alpha}}. \quad (4)$$

By  $2\pi$ -periodicity of  $\widehat{h}_j$  this inequality can be extended to  $\lambda \in \mathbb{R}$  as

$$|\widehat{h}_j(\lambda)| \leq C \frac{\gamma_j^{1/2} |\gamma_j \{\lambda\}|^M}{(1 + \gamma_j |\{\lambda\}|)^{\alpha+M}}. \quad (5)$$

where  $\{\lambda\}$  denotes the element of  $(-\pi, \pi]$  such that  $\lambda - \{\lambda\} \in 2\pi\mathbb{Z}$ .

- c. Asymptotic behavior: There exists some non identically zero function  $\widehat{h}_\infty$  such that for any  $\lambda \in \mathbb{R}$ ,

$$\lim_{j \rightarrow +\infty} (\gamma_j^{-1/2} \widehat{h}_j(\gamma_j^{-1} \lambda)) = \widehat{h}_\infty(\lambda). \quad (6)$$

Observe that while  $\widehat{h}_j$  is  $2\pi$ -periodic, the function  $\widehat{h}_\infty$  is a non-periodic function on  $\mathbb{R}$  (this follows from (12) below). For the connection between these assumptions on  $h_j$  and corresponding assumptions on the scaling function  $\varphi$  and the mother wavelet  $\psi$  in the classical wavelet setting (2) (see [1]). In particular, in that case, one has  $\widehat{h}_\infty = \widehat{\varphi}(0) \widehat{\psi}$ .

Our goal is to study the large scale behavior of the random wavelet coefficients

$$W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j(\gamma_j k - \ell) Y_\ell = \sum_{\ell \in \mathbb{Z}} h_j(\gamma_j k - \ell) (\Delta^{-K} G(X))_\ell, \quad (7)$$

where we set symbolically  $Y_\ell = (\Delta^{-K} G(X))_\ell$  for  $(\Delta^K Y)_\ell = G(X_\ell)$ .

By Assumption (4),  $h_j$  has null moments up to order  $M - 1$ , that is, for any  $m \in \{0, \dots, M - 1\}$ ,

$$\sum_{\ell \in \mathbb{Z}} h_j(\ell) \ell^m = 0. \quad (8)$$

Therefore, since  $M \geq K$ ,  $\widehat{h}_j$  can be expressed as

$$\widehat{h}_j(\lambda) = (1 - e^{-i\lambda})^K \widehat{h}_j^{(K)}(\lambda), \quad (9)$$

where  $\widehat{h}_j^{(K)}$  is also a trigonometric polynomial of the form

$$\widehat{h}_j^{(K)}(\lambda) = \sum_{\tau \in \mathbb{Z}} h_j^{(K)}(\tau) e^{-i\lambda\tau}, \quad (10)$$

since  $h_j^{(K)}$  has finite support for any  $j$ . Then we obtain another way of expressing  $W_{j,k}$ , namely,

$$W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) G(X_\ell). \quad (11)$$

We have thus incorporated the linear filter  $\Delta^{-K}$  in (7) into the filter  $h_j$  and denoted the new filter  $h_j^{(K)}$ .

### Remarks

1. Since  $\{G(X_\ell), \ell \in \mathbb{Z}\}$  is stationary, it follows from (11) that  $\{W_{j,k}, k \in \mathbb{Z}\}$  is stationary for each scale  $j$ .
2. Observe that  $\Delta^K Y$  is centered by definition. However, by (8), the definition of  $W_{j,k}$  only depends on  $\Delta^M Y$ . In particular, provided that  $M \geq K + 1$ , its value is not modified if a constant is added to  $\Delta^K Y$ , whenever  $M \geq K + 1$ .

3. Assumptions (4) and (6) imply that for any  $\lambda \in \mathbb{R}$ ,

$$|\widehat{h}_\infty(\lambda)| \leq C \frac{|\lambda|^M}{(1 + |\lambda|)^{\alpha+M}}. \quad (12)$$

Hence  $\widehat{h}_\infty \in L^2(\mathbb{R})$  since  $\alpha > 1/2$ .

4. The Fourier transform of  $f$ ,

$$\mathfrak{F}(f)(\xi) = \int_{\mathbb{R}^q} f(t) e^{-it^T \xi} dt, \quad \xi \in \mathbb{R}^q, \quad (13)$$

is defined for any  $f \in L^2(\mathbb{R}^q, \mathbb{C})$ . We let  $h_\infty$  be the  $L^2(\mathbb{R})$  function such that  $\widehat{h}_\infty = \mathfrak{F}[h_\infty]$ .

### 3. Integral representations

It is convenient to use an integral representation in the spectral domain to represent the random processes (see for example [17, 18]). The stationary Gaussian process  $\{X_k, k \in \mathbb{Z}\}$  with spectral density (1) can be written as

$$X_\ell = \int_{-\pi}^{\pi} e^{i\lambda \ell} f^{1/2}(\lambda) d\widehat{W}(\lambda) = \int_{-\pi}^{\pi} \frac{e^{i\lambda \ell} f^{*1/2}(\lambda)}{|1 - e^{-i\lambda}|^d} d\widehat{W}(\lambda), \quad \ell \in \mathbb{N}. \quad (14)$$

This is a special case of

$$\widehat{I}(g) = \int_{\mathbb{R}} g(x) d\widehat{W}(x), \quad (15)$$

where  $\widehat{W}(\cdot)$  is a complex-valued Gaussian random measure satisfying

$$\mathbb{E}(\widehat{W}(A)) = 0 \quad \text{for every Borel set } A \text{ in } \mathbb{R}, \quad (16)$$

$$\mathbb{E}(\widehat{W}(A) \overline{\widehat{W}(B)}) = |A \cap B| \quad \text{for every Borel sets } A \text{ and } B \text{ in } \mathbb{R}, \quad (17)$$

$$\sum_{j=1}^n \widehat{W}(A_j) = \widehat{W}\left(\bigcup_{j=1}^n A_j\right) \quad \text{if } A_1, \dots, A_n \text{ are disjoint Borel sets in } \mathbb{R}, \quad (18)$$

$$\widehat{W}(A) = \overline{\widehat{W}(-A)} \quad \text{for every Borel set } A \text{ in } \mathbb{R}. \quad (19)$$

The integral (15) is defined for any function  $g \in L^2(\mathbb{R})$  and one has the isometry

$$\mathbb{E}(|\widehat{I}(g)|^2) = \int_{\mathbb{R}} |g(x)|^2 dx.$$

The integral  $\widehat{I}(g)$ , moreover, is real-valued if

$$g(x) = \overline{g(-x)}.$$

We shall also consider multiple Itô–Wiener integrals

$$\widehat{I}_q(g) = \int_{\mathbb{R}^q}'' g(\lambda_1, \dots, \lambda_q) d\widehat{W}(\lambda_1) \cdots d\widehat{W}(\lambda_q)$$

where the double prime indicates that one does not integrate on hyperdiagonals  $\lambda_i = \pm\lambda_j, i \neq j$ . The integrals  $\widehat{I}_q(g)$  are handy because we will be able to expand our non-linear functions  $G(X_k)$  introduced in Section 1 in multiple integrals of this type.

These multiples integrals are defined as follows. Denote by  $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$  the space of complex valued functions defined on  $\mathbb{R}^q$  satisfying

$$g(-x_1, \dots, -x_q) = \overline{g(x_1, \dots, x_q)} \text{ for } (x_1, \dots, x_q) \in \mathbb{R}^q, \quad (20)$$

$$\|g\|_{L^2}^2 := \int_{\mathbb{R}^q} |g(x_1, \dots, x_q)|^2 dx_1 \cdots dx_q < \infty. \quad (21)$$

Let  $\tilde{L}^2(\mathbb{R}^q, \mathbb{C})$  denote the set of functions in  $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$  that are symmetric in the sense that  $g = \tilde{g}$  where  $\tilde{g}(x_1, \dots, x_q) = 1/q! \sum_{\sigma} g(x_{\sigma(1)}, \dots, x_{\sigma(q)})$ , where the sum is over all permutations of  $\{1, \dots, q\}$ . One defines now the multiple integral with respect to the spectral measure  $\widehat{W}$  by a density argument. For a step function of the form

$$g = \sum_{j_\ell = \pm 1, \dots, \pm N} c_{j_1, \dots, j_n} 1_{\Delta_{j_1}} \times \cdots \times 1_{\Delta_{j_n}}$$

where the  $c$ 's are real-valued,  $\Delta_{j_\ell} = -\Delta_{-j_\ell}$  and  $\Delta_{j_\ell} \cap \Delta_{j_m} = \emptyset$  if  $\ell \neq m$ , one sets

$$\widehat{I}_q(g) = \sum_{j_\ell = \pm 1, \dots, \pm N}'' c_{j_1, \dots, j_n} \widehat{W}(\Delta_{j_1}) \cdots \widehat{W}(\Delta_{j_n}). \quad (22)$$

Here,  $\sum''$  indicates that one does not sum over the hyperdiagonals, that is, when  $j_\ell = \pm j_m$  for  $\ell \neq m$ . The integral  $\widehat{I}_q$  verifies that

$$\mathbb{E}(\widehat{I}_q(g_1) \widehat{I}_{q'}(g_2)) = \begin{cases} q! \langle g_1, g_2 \rangle_{L^2}, & \text{if } q = q' \\ 0, & \text{if } q \neq q'. \end{cases} \quad (23)$$

Observe, moreover, that for every step function  $g$  with  $q$  variables as above

$$\widehat{I}_q(g) = \widehat{I}_q(\tilde{g}).$$

Since the set of step functions is dense in  $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$ , one can extend  $\widehat{I}_q$  to an isometry from  $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$  to  $L^2(\Omega)$  and the above properties hold true for this extension.

**Remark.** Property (20) of the function  $f$  in  $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$  together with Property (19) of  $\widehat{W}$  ensure that  $\widehat{I}_q(f)$  is a real-valued random variable.

#### 4. Wiener Chaos

Our results are based on the expansion of the function  $G$ , introduced in Section 1, in Hermite polynomials. The Hermite polynomials are

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} \left( e^{-\frac{x^2}{2}} \right) ,$$

in particular,  $H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1$ . If  $X$  is a normal random variable with mean 0 and variance 1, then

$$\mathbb{E}(H_q(X)H_{q'}(X)) = \int_{\mathbb{R}} H_q(x)H_{q'}(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = q! \delta_{q,q'} .$$

Moreover,

$$G(X) = \sum_{q=1}^{+\infty} \frac{c_q}{q!} H_q(X) , \quad (24)$$

where the convergence is in  $L^2(\Omega)$  and where

$$c_q = \mathbb{E}(G(X)H_q(X)) . \quad (25)$$

The expansion (24) is called a Wiener chaos expansion with each term in the chaos expansion living in a different chaos. The expansion (24) starts at  $q = 1$ , since

$$c_0 = \mathbb{E}(G(X)H_0(X)) = \mathbb{E}(G(X)) = 0 ,$$

by assumption. The condition  $\mathbb{E}(G(X)^2) < \infty$  implies

$$\sum_{q=1}^{+\infty} \frac{c_q^2}{q!} < \infty . \quad (26)$$

Hermite polynomials are related to multiple integrals as follows : if  $X = \int_{\mathbb{R}} g(x) d\widehat{W}(x)$  with  $\mathbb{E}(X^2) = \int_{\mathbb{R}} |g(x)|^2 dx = 1$  and  $g(x) = \overline{g(-x)}$  so that  $X$  has unit variance and is real-valued, then

$$H_q(X) = \widehat{I}_q(g^{\otimes q}) = \int_{\mathbb{R}^q} g(x_1) \cdots g(x_q) d\widehat{W}(x_1) \cdots d\widehat{W}(x_q) . \quad (27)$$

The expansion (24) of  $G$  induces a corresponding expansion of the wavelet coefficients  $W_{j,k}$ , namely,

$$W_{j,k} = \sum_{q=1}^{+\infty} \frac{c_q}{q!} W_{j,k}^{(q)} , \quad (28)$$

where by (11) one has

$$W_{j,k}^{(q)} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) H_q(X_\ell) . \quad (29)$$



The Gaussian sequence  $\{X_n\}_{n \in \mathbb{Z}}$  is long-range dependent because its spectrum at low frequencies behaves like  $|\lambda|^{-2d}$  with  $d > 0$  and hence explodes at  $\lambda = 0$ . What about the processes  $\{H_q(X_\ell)\}_\ell$  for  $q \geq 2$ ? What is the behavior of the spectrum at low frequencies? Does it explodes at  $\lambda = 0$ ? The answer depends on the respective values of  $q$  and  $d$ . Let us define

$$q_c = \max\{q \in \mathbb{N} : q < 1/(1 - 2d)\}, \quad (30)$$

and

$$d(q) = qd + (1 - q)/2. \quad (31)$$

One has

$$d(q) > 0 \quad \text{if } q \leq q_c, \quad \text{that is if } q < 1/(1 - 2d). \quad (32)$$

The following result shows that the spectral density of  $\{H_q(X_\ell)\}_{\ell \in \mathbb{Z}}$  has a different behavior at zero frequency depending on whether  $q \leq q_c$  or  $q > q_c$ . It is long-range dependent when  $q \leq q_c$  and short-range dependent when  $q > q_c$ . We first give a definition.

**Definition 4.1.** *The convolution of two locally integrable  $(2\pi)$ -periodic functions  $g_1$  and  $g_2$  is defined as*

$$(g_1 \star g_2)(\lambda) = \int_{-\pi}^{\pi} g_1(u)g_2(\lambda - u)du. \quad (33)$$

Moreover the  $q$  times self-convolution of  $g$  is denoted by  $g^{(\star q)}$ .

**Lemma 4.1.** *Let  $q$  be a positive integer. The spectral density of  $\{H_q(X_\ell)\}_{\ell \in \mathbb{Z}}$  is*

$$q!f^{(\star q)} = q!(f \star \cdots \star f),$$

where the spectral density  $f$  of  $\{X_\ell\}_{\ell \in \mathbb{Z}}$  is given in (1). Moreover the following holds :

- (i) *If  $q \leq q_c$ , then  $\lambda^{2d(q)}f^{(\star q)}(\lambda)$  is bounded on  $\lambda \in (0, \pi)$  and converges to a positive number as  $\lambda \downarrow 0$ .*
- (ii) *If  $q > q_c$ , then  $f^{(\star q)}(\lambda)$  is bounded on  $\lambda \in (0, \pi)$  and converges to a positive number as  $\lambda \downarrow 0$ .*

Hence if  $q \leq q_c$ ,  $\{H_q(X_\ell)\}_\ell$  has long memory with parameter  $d(q) > 0$  whereas if  $q > q_c$ ,  $\{H_q(X)\}_\ell$  has a short-memory behavior.

**Proof.** By definition of  $H_q$  and since  $X$  has unit variance by assumption, we have

$$\mathbb{E}(H_q(X_\ell)H_q(X_{\ell+m})) = q! \left( \int_{-\pi}^{\pi} f(\lambda)e^{i\lambda m}d\lambda \right)^q.$$

Using the fact that, for any two locally integrable  $(2\pi)$ -periodic functions  $g_1$  and  $g_2$ , one has

$$\int_{-\pi}^{\pi} (g_1 \star g_2)(\lambda)e^{i\lambda m}d\lambda = \int_{-\pi}^{\pi} g_1(u)e^{i u m}du \times \int_{-\pi}^{\pi} g_2(v)e^{i v m}dv,$$

we obtain that the spectral density of  $\{H_q(X_\ell)\}_\ell$  is  $q!f^{(\star q)}$ .

The properties of  $f^{(\star q)}$  stated in Lemma 4.1 are proved by induction on  $q$  using Lemma 8.2. Observe indeed that if  $\beta_1 = d(q)$  and  $\beta_2 = 2d$ , then

$$\beta_1 + \beta_2 - 1 = 2d(q) + 2d - 1 = (2dq + 1 - q) + 2d - 1 = 2(q + 1)d - (q + 1) + 1 = 2d(q + 1) .$$

□

Now, consider the expansion of  $\Delta^K Y_\ell = G(X_\ell) = \sum_{q=q_0}^{+\infty} (c_q/q!)H_q(X_\ell)$ , where

$$q_0 = \min\{q \geq 1, c_q \neq 0\} . \quad (34)$$

The exponent  $q_0$  is called the *Hermite rank* of  $\Delta^K Y$ .

In the following, we always assume that at least one summand of  $\Delta^K Y_\ell$  has long memory, that is, in view of Lemma 4.1,

$$q_0 \leq q_c . \quad (35)$$

## 5. The result and its interpretations

In this section we describe the limit in distribution of the wavelet coefficients  $\{W_{j+m,k}\}_{m,k}$  as  $j \rightarrow \infty$ , adequately normalized, and we interpret the limit. Recall that  $W_{j+m,k}$  involves a sum of chaoses of all order. In the limit, however, only the order  $q_0$  will prevail. The convergence of finite-dimensional distributions is denoted by  $\xrightarrow{\text{fidi}}$ .

**Theorem 5.1.** *As  $j \rightarrow \infty$ , we have*

$$\left\{ \gamma_j^{-(d(q_0)+K)} W_{j+m,k}, m, k \in \mathbb{Z} \right\} \xrightarrow{\text{fidi}} c_{q_0} (f^*(0))^{q_0/2} \left\{ Y_{m,k}^{(q_0,K)}, m, k \in \mathbb{Z} \right\} , \quad (36)$$

where for every positive integer  $q$ ,

$$Y_{m,k}^{(q,K)} = (\bar{\gamma}_m)^{1/2} \int_{\mathbb{R}^q} \frac{e^{ik\bar{\gamma}_m(\zeta_1+\dots+\zeta_q)}}{(i(\zeta_1+\dots+\zeta_q))^K} \frac{\widehat{h}_\infty(\bar{\gamma}_m(\zeta_1+\dots+\zeta_q))}{|\zeta_1|^d \dots |\zeta_q|^d} d\widehat{W}(\zeta_1) \dots d\widehat{W}(\zeta_q) . \quad (37)$$

This Theorem is proved in Section 6.

### Interpretation of the limit.

The limit distribution can be interpreted as the wavelet coefficients of a generalized Hermite process defined below, based on the wavelet family

$$\{h_{\infty,m,k}(t) = \bar{\gamma}_m^{-1/2} h_\infty(-\bar{\gamma}_m^{-1}t + k), m, k \in \mathbb{Z}\} . \quad (38)$$

This wavelet family is the natural one to consider because the Fourier transform  $\widehat{h}_\infty(\lambda)$  is the rescaled limit of the original  $\widehat{h}_j(\lambda)$  as indicated in (6).

A generalized process is indexed not by time but by functions. The generalized Hermite processes for any order  $q$  in  $\{1, \dots, q_c\}$  are defined as follows :

**Definition 5.1.** Let  $0 < d < 1/2$  and let  $q$  be a positive integer such that  $0 < q < 1/(1-2d)$  and  $K \geq 0$ . Define the set of functions

$$\mathcal{S}_{q,d}^{(K)} = \left\{ \theta, \int_{\mathbb{R}} |\widehat{\theta}(\xi)|^2 |\xi|^{q-1-2dq-2K} d\xi < \infty \right\},$$

where  $\widehat{\theta} = \mathfrak{F}[\theta]$ . The generalized random process  $Z_{q,d}^{(K)}$  is indexed by functions  $\theta \in \mathcal{S}_{q,d}^{(K)}$  and is defined as

$$Z_{q,d}^{(K)}(\theta) = \int_{\mathbb{R}^q} \frac{\overline{\widehat{\theta}(u_1 + \dots + u_q)}}{(i(u_1 + \dots + u_q))^K |u_1 \dots u_q|^d} d\widehat{W}(u_1) \dots d\widehat{W}(u_q), \quad (39)$$

where  $\widehat{\theta} = \mathfrak{F}[\theta]$  as defined in (13).

Now fix  $(m, k) \in \mathbb{Z}^2$  and choose a function  $h_{\infty,m,k}(t), t \in \mathbb{R}$  as in (38), so that

$$\mathfrak{F}[h_{\infty,m,k}](\xi) = \mathfrak{F}[\overline{\gamma_m^{-1/2} h_{\infty}(-\overline{\gamma_m^{-1} t + k})}](\xi) = (\overline{\gamma_m})^{1/2} e^{-i\overline{\gamma_m} \xi} \overline{\widehat{h_{\infty}(\overline{\gamma_m} \xi)}}. \quad (40)$$

**Lemma 5.1.** The conditions on  $d$  and  $q$  in Definition 5.1 ensures the existence of  $Z_{q,d}^{(K)}(\theta)$ . In particular,

$$h_{\infty,m,k} \in \mathcal{S}_{q,d}^{(K)} \text{ for all } K \in \{0, \dots, M\},$$

and hence  $Z_{q,d}^{(K)}(h_{\infty,m,k})$  is well-defined.

This Lemma is proved in Section 7.

By setting in (39),  $\theta = h_{\infty,m,k}$ , defined in (40), we obtain for all  $(m, k) \in \mathbb{Z}^2$ ,

$$Y_{m,k}^{(q,K)} = Z_{q,d}^{(K)}(h_{\infty,m,k}).$$

Hence the right-hand side of (36) are the wavelet coefficients of the generalized process  $Z_{q,d}^{(K)}$  with respect to the wavelet family  $\{h_{\infty,m,k}, m, k \in \mathbb{Z}\}$ .

In the special case  $q = 1$  (Gaussian case), this result corresponds to that of Theorem 1(b) and Remark 5 in [1], obtained in the case where  $\gamma_j = 2^j$ . In this special case, we have  $Z_{1,d}^{(K)} = B_{(d+K)}$ , where  $B_{(d)}$  is the centered generalized Gaussian process such that for all  $\theta_1, \theta_2 \in \mathcal{S}_{1,d}^{(0)}$ ,

$$\text{Cov}(B_{(d)}(\theta_1), B_{(d)}(\theta_2)) = \int_{\mathbb{R}} |\lambda|^{-2d} \widehat{\theta}_1(\lambda) \overline{\widehat{\theta}_2(\lambda)} d\lambda.$$

It is interesting to observe that, under additional assumptions on  $\theta$ , for  $K \geq 1$ ,  $Z_{q,d}^{(K)}(\theta)$  can also be defined by

$$Z_{q,d}^{(K)}(\theta) = \int_{\mathbb{R}} \tilde{Z}_{q,d}^{(K)}(t) \overline{\theta(t)} dt, \quad (41)$$

where  $\{\tilde{Z}_{q,d}^{(K)}(t), t \in \mathbb{R}\}$  denotes a measurable continuous time process defined by

$$\tilde{Z}_{q,d}^{(K)}(t) = \int_{\mathbb{R}^q} \frac{e^{i(u_1 + \dots + u_q)t} - \sum_{\ell=0}^{K-1} \frac{(i(u_1 + \dots + u_q)t)^\ell}{\ell!}}{(i(u_1 + \dots + u_q))^K |u_1 \dots u_q|^d} d\widehat{W}(u_1) \dots d\widehat{W}(u_q), t \in \mathbb{R}. \quad (42)$$

If, in (41) we set  $K = 1$ , we recover the usual Hermite process as defined in [19] which has stationary increments. The process  $\tilde{Z}_{q,d}^{(K)}(t)$  can be regarded as the Hermite process  $\tilde{Z}_{q,d}^{(1)}(t)$  integrated  $K - 1$  times. In the special case where  $K = q = 1$ , we recover the Fractional Brownian Motion  $\{B_H(t)\}_{t \in \mathbb{R}}$  with Hurst index  $H = d + 1/2 \in (1/2, 1)$ .

In the case  $K = 0$  we cannot define a random process  $Z_{q,d}^{(0)}(t)$  as in (42). The case  $K = 0$  would correspond to the derivative of the Hermite process  $\tilde{Z}_{q,d}^{(1)}(t)$  but the Hermite process is not differentiable and thus the process  $\tilde{Z}_{q,d}^{(0)}(t), t \in \mathbb{R}$  is not defined. When  $K = 0$  one can only consider the generalized process  $\tilde{Z}_{q,d}^{(0)}(\theta)$ . Relation (42) can be viewed as resulting from (39) and (41) by interverting formally the integral signs.

We now state sufficient conditions on  $\theta$  for (41) to hold.

**Lemma 5.2.** *Let  $q$  be a positive integer such that  $0 < q < 1/(1 - 2d)$  and  $K \geq 1$ . Suppose that  $\theta \in \mathcal{S}_{q,\delta}^{(K)}$  is complex valued with at least  $K$  vanishing moments, that is,*

$$\int_{\mathbb{R}} \theta(t) t^\ell dt = 0 \quad \text{for all } \ell = 0, 1, \dots, K - 1. \quad (43)$$

Suppose moreover that

$$\int_{\mathbb{R}} |\theta(t)| |t|^{K+(d-1/2)q} dt < \infty. \quad (44)$$

Then Relation (41) holds.

This lemma is proved in Section 7.

If, for example, the  $h_j$  are derived from a compactly supported multiresolution analysis then  $h_\infty$  will have compact support and so  $h_{\infty,m,k}$  will satisfy (44). In this case, the limits  $Y_{m,k}^{(q,K)}$  in Theorem 5.1 can therefore be interpreted, for  $m, k \in \mathbb{Z}$  as the wavelet coefficients of the process  $Z_{q,d}^{(K)}$  belonging to the  $q$ -th chaos. This interpretation is a useful one even when the technical assumption (44) is not satisfied.

### Self-similarity.

The processes  $Z_{q,d}^{(K)}$  and  $\tilde{Z}_{q,d}^{(K)}$  are self-similar. Self-similarity can be defined for processes indexed by  $t \in \mathbb{R}$  as well as for generalized processes indexed by functions  $\theta$  belonging to some suitable space  $\mathcal{S}$ , for example the space  $\mathcal{S}_{q,\delta}^{(K)}$  defined above.

A process  $\{Z(t), t \in \mathbb{R}\}$  is said to be *self-similar with parameter  $H > 0$*  if for any  $a > 0$ ,

$$\{a^H Z(t/a), t \in \mathbb{R}\} \stackrel{\text{fdi}}{=} \{Z(t), t \in \mathbb{R}\},$$

where the equality holds in the sense of finite-dimensional distributions. A generalized process  $\{Z(\theta), \theta \in \mathcal{S}\}$  is said to be *self-similar with parameter  $H > 0$*  if for any  $a > 0$  and  $\theta \in \mathcal{S}$ ,

$$Z(\theta^{a,H}) \stackrel{\text{d}}{=} Z(\theta),$$

where  $\theta^{a,H}(u) = a^{-H}\theta(u/a)$  (see [17], Page 5). Here  $\mathcal{S}$  is assumed to contain both  $\theta^{a,H}$  and  $\theta$ .

Observe that the process  $\{\tilde{Z}_{q,d}^{(K)}(t), t \in \mathbb{R}\}$ , with  $K \geq 1$  is self-similar with parameter

$$H = K + qd - q/2 = (K - 1) + (d(q) + 1/2). \quad (45)$$

As noted above  $\tilde{Z}_{q,d}^{(K)}$  can be regarded as  $\tilde{Z}_{q,d}^{(1)}$  integrated  $K - 1$  times.

The generalized process  $\{Z_{q,d}^{(K)}(\theta), \theta \in \mathcal{S}_{q,d}^{(K)}\}$ , which is defined in (39) with  $K \geq 0$ , is self-similar with the same value of  $H$  as in (45), but this time the formula is also valid for  $K = 0$ .

In particular, the Hermite process ( $K = 1$ ) is self-similar with  $H = d(q) + 1/2 \in (1/2, 1)$  and the generalized process  $Z_{q,d}^{(0)}(\theta)$  with  $K = 0$  is self-similar with  $H = d(q) - 1/2 \in (-1/2, 0)$ .

### Interpretation of the result.

In view of the preceding discussion, the wavelet coefficients of the subordinated process  $Y$  behave at large scales ( $\gamma_j \rightarrow \infty$ ) as those of a self-similar process  $Z_{q,d}^{(K)}$  living in the chaos of order  $q_0$  (the Hermite rank of  $G$ ) and with self-similar parameter  $K + d(q_0) - 1/2$ .

## 6. Proof of Theorem 5.1

**Notation.** It will be convenient to use the following notation. We denote by  $\Sigma_q$ ,  $q \geq 1$ , the  $\mathbb{C}^q \rightarrow \mathbb{C}$  function defined, for all  $y = (y_1, \dots, y_q)$  by

$$\Sigma_q(y) = \sum_{i=1}^q y_i. \quad (46)$$

With this notation  $Y_{m,k}^{(q,K)}$  in Theorem 5.1 can be expressed as

$$Y_{m,k}^{(q,K)} = (\bar{\gamma}_m)^{1/2} \int_{\mathbb{R}^q}'' \frac{\exp \circ \Sigma_q(ik\bar{\gamma}_m\zeta)}{(\Sigma_q(i\zeta))^K} \cdot \frac{\hat{h}_\infty \circ \Sigma_q(\bar{\gamma}_m\zeta)}{|\zeta_1|^d \dots |\zeta_q|^d} d\widehat{W}(\zeta_1) \dots d\widehat{W}(\zeta_q).$$

where  $\circ$  denotes the composition of functions.

We will separate the Wiener chaos expansion (28) of  $W_{j,k}$  into two terms depending on the position of  $q$  with respect to  $q_c$ . The first term includes only the  $q$ 's for which  $H_q(x)$  exhibits long-range dependence (LD), that is,

$$W_{j,k}^{(LD)} = \sum_{q=0}^{q_c} \frac{c_q}{q!} W_{j,k}^{(q)}, \quad (47)$$

and the second term includes the terms which exhibit short-range dependence (SD)

$$W_{j,k}^{(SD)} = \sum_{q=q_c+1}^{\infty} \frac{c_q}{q!} W_{j,k}^{(q)}. \quad (48)$$

Using Representation (14) and (27) since  $X$  has unit variance, one has for any  $\ell \in \mathbb{Z}$ ,

$$\begin{aligned} H_q(X_\ell) &= H_q \left( \int_{-\pi}^{\pi} e^{i\xi\ell} f^{1/2}(\xi) d\widehat{W}(\xi) \right) \\ &= \int_{(-\pi, \pi]^q}'' \exp \circ \Sigma_q(i\ell\xi) \times (f^{\otimes q}(\xi))^{1/2} d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q). \end{aligned}$$

Then by (29),(10) and (9), we have

$$\begin{aligned} W_{j,k}^{(q)} &= \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) H_q(X_\ell) \\ &= \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) \int_{(-\pi, \pi]^q}'' \exp \circ \Sigma_q(i\ell\xi) \times (f^{\otimes q}(\xi))^{1/2} d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q) \\ &= \int_{(-\pi, \pi]^q}'' \left( \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) \exp \circ \Sigma_q(i\ell\xi) \right) (f^{\otimes q}(\xi))^{1/2} d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q) \\ &= \int_{(-\pi, \pi]^q}'' e^{\Sigma_q(i\gamma_j k \xi)} \left( \sum_{m \in \mathbb{Z}} h_j^{(K)}(m) \exp \circ \Sigma_q(-im\xi) \right) (f^{\otimes q}(\xi))^{1/2} d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q) \\ &= \int_{(-\pi, \pi]^q}'' e^{\Sigma_q(i\gamma_j k \xi)} \left( \widehat{h}_j^{(K)} \circ \Sigma_q(\xi) \right) (f^{\otimes q}(\xi))^{1/2} d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q). \end{aligned}$$

Then

$$W_{j,k}^{(q)} = \widehat{I}_q(f_{j,k}^{(q)}), \quad (49)$$

with

$$f_{j,k}^{(q)}(\xi) = (\exp \circ \Sigma_q(ik\gamma_j \xi)) \left( \widehat{h}_j^{(K)} \circ \Sigma_q(\xi) \right) (f^{\otimes q}(\xi))^{1/2} \times \mathbb{1}_{(-\pi, \pi)}^{\otimes q}(\xi),$$

where  $\xi = (\xi_1, \dots, \xi_q)$  and  $f^{\otimes q}(\xi) = f(\xi_1) \cdots f(\xi_q)$ .

The two following results provide the asymptotic behavior of each term of the sum in (47) and of  $W_{j,k}^{(SD)}$ , respectively. They are proved in Sections 6.1 and 6.2, respectively. The first result concerns the terms with long memory, that is, with  $q \leq q_c$ . The second result concerns the terms with short memory for which  $q > q_c$ .

**Proposition 6.1.** *Suppose that  $q \in \{1, \dots, q_c\}$ . Then, as  $j \rightarrow \infty$ ,*

$$\left( \gamma_j^{-(d(q)+K)} W_{j+m,k}^{(q)}, m, k \in \mathbb{Z} \right) \xrightarrow{fidi} \left( (f^*(0))^{q/2} Y_{m,k}^{(q,K)}, m, k \in \mathbb{Z} \right), \quad (50)$$

where  $Y_{m,k}^{(q,K)}$  is given by (37).

**Proposition 6.2.** *We have, for any  $k \in \mathbb{Z}$ , as  $j \rightarrow \infty$ ,*

$$W_{j+m,k}^{(SD)} = O_P(\gamma_j^K). \quad (51)$$

It follows from Proposition 6.1 that the dominating term in (47) is given by the chaos of order  $q = q_0$ . Now, since  $d(q_0) > 0$  by (32), we get from Proposition 6.2 that, for all  $(k, m)$ , as  $j \rightarrow \infty$ ,

$$W_{j+m,k}^{(SD)} = o_P(\gamma_j^{d(q_0)+K}).$$

This concludes the proof of Theorem 5.1.

### 6.1. Proof of Proposition 6.1

We first express the distribution of  $\{W_{j+m,k}^{(q)}, m, k \in \mathbb{Z}\}$  as a finite sum of stochastic integrals and then show that each integral converges in  $L^2(\Omega)$ .

**Lemma 6.1.** *Let  $q \in \mathbb{N}^*$ . For any  $j$*

$$W_{j+m,k}^{(q)} \stackrel{(fidi)}{=} \sum_{s=-[q/2]}^{[q/2]} W_{m,k}^{(j,q,s)}, \quad (52)$$

where  $[a]$  denotes the integer part of  $a$ , and for any  $q \in \mathbb{N}^*$ ,  $s \in \mathbb{Z}$ ,

$$W_{m,k}^{(j,q,s)} = \int_{\zeta \in \mathbb{R}^q}'' \mathbb{1}_{\Gamma^{(q,s)}}(\gamma_j^{-1}\zeta) f_{m,k}(\zeta; j, q) d\widehat{W}(\zeta_1) \cdots d\widehat{W}(\zeta_q), \quad (53)$$

where  $f_{m,k}(\zeta; j, q)$  is defined by (setting  $\xi = \gamma_j^{-1}\zeta$ )

$$f_{m,k}(\gamma_j\xi; j, q) = \gamma_j^{-q/2} \frac{\exp \circ \Sigma_q(i\gamma_{j+m}k\xi) \times \widehat{h}_{j+m} \circ \Sigma_q(\xi)}{\{1 - \exp \circ \Sigma_q(-i\xi)\}^K} (f^{\otimes q}(\xi))^{1/2}. \quad (54)$$

and where

$$\Gamma^{(q,s)} = \left\{ \xi \in (-\pi, \pi]^q, -\pi + 2s\pi < \sum_{i=1}^q \xi_i \leq \pi + 2s\pi \right\}. \quad (55)$$

**Proof.** Using (49), with  $j$  replaced by  $j + m$ , and (9), we get

$$W_{j+m,k}^{(q)} = \int_{(-\pi, \pi]^q}'' \exp \circ \Sigma_q(i\gamma_{j+m}k\xi) \frac{\widehat{h}_{j+m} \circ \Sigma_q(\xi)}{\{1 - \exp \circ \Sigma_q(i\xi)\}^K} (f^{\otimes q}(\xi))^{1/2} d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q).$$

By (54), we thus get

$$\begin{aligned} W_{j+m,k}^{(q)} &= \int_{\xi \in (-\pi, \pi]^q}'' \gamma_j^{q/2} f_{m,k}(\gamma_j\xi; j, q) d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q) \\ &\stackrel{(fidi)}{=} \int_{\zeta \in (-\gamma_j\pi, \gamma_j\pi]^q}'' f_{m,k}(\zeta; j, q) d\widehat{W}(\zeta_1) \cdots d\widehat{W}(\zeta_q), \end{aligned} \quad (56)$$

where we set  $\zeta = \gamma_j \xi$  (see Theorem 4.4 in [17]). Observe that for all  $\zeta \in (-\gamma_j \pi, \gamma_j \pi]^q$ ,

$$-\pi \gamma_j - 2[q/2] \pi \gamma_j \leq -q \gamma_j \pi \leq \sum_{i=1}^q \zeta_i \leq q \gamma_j \pi \leq \pi \gamma_j + 2[q/2] \pi \gamma_j .$$

The result follows by using that for any  $\zeta \in (-\gamma_j \pi, \gamma_j \pi]^q$ , there is a unique  $s = -[q/2], \dots, [q/2]$  such that  $\zeta/\gamma_j \in \Gamma^{(q,s)}$ .  $\square$

**Proof of Proposition 6.1.** In view of Lemma 6.1, we shall look at the  $L^2(\Omega)$  convergence of the normalized  $W_{m,k}^{(j,q,s)}$  at each value of  $s$ . Proposition 6.1 will follow from the following convergence results, valid for all fixed  $m, k \in \mathbb{Z}$  as  $j \rightarrow \infty$ . For  $s = 0$ ,

$$\gamma_j^{-(d(q)+K)} W_{m,k}^{(j,q,0)} \xrightarrow{L^2} (f^*(0))^{q/2} Y_{m,k}^{(q,K)} , \quad (57)$$

whereas for other values of  $s$ , namely for all  $s \in \{-[q/2], \dots, -1, 1, \dots, [q/2]\}$ ,

$$\gamma_j^{-(d(q)+K)} W_{m,k}^{(j,q,s)} \xrightarrow{L^2} 0 , \quad (58)$$

where  $d(q)$  is defined in (31).

We now prove these convergence using the representation (53). By (1) and  $|1 - e^{i\lambda}| \geq 2|\lambda|/\pi$  on  $\lambda \in (-\pi, \pi)$ , we have that

$$f(\lambda) \leq \left(\frac{\pi}{2}\right)^{-2d} \|f^*\|_\infty |\lambda|^{-2d} , \quad \lambda \in [-\pi, \pi] . \quad (59)$$

By definition of  $\Gamma^{(q,s)}$  in (55), we have, for all  $\zeta \in \gamma_j \Gamma^{(q,s)}$ ,  $\gamma_j^{-1} \sum_i \zeta_i - 2\pi s \in (-\pi, \pi]$ . Hence using the  $(2\pi)$ -periodicity of  $\widehat{h}_{j+m}$ , we can use (4) for bounding  $\widehat{h}_{j+m}(\gamma_j^{-1} \sum_i \zeta_i)$ . With the change of variables  $\zeta = \gamma_j \xi$  and (59), for all  $\zeta \in \gamma_j \Gamma^{(q,s)}$  and  $j$  large enough so that  $\gamma_{j+m}/\gamma_j \geq \overline{\gamma}_m/2$ ,

$$\gamma_j^{-(d(q)+K)} |f_{m,k}(\zeta; j, q)| = \gamma_j^{-(dq-q/2+1/2+K)} |f_{m,k}(\zeta; j, q)| \leq C_0 g(\zeta; 2\pi \gamma_j s) , \quad (60)$$

where  $C_0$  is a positive constant and

$$g(\zeta; t) = \left(1 + \left| \sum_{i=1}^q \zeta_i - t \right| \right)^{-\alpha-K} \prod_{i=1}^q |\zeta_i|^{-d} .$$

The squared  $L^2$ -norm of  $g(\cdot; t)$  reads

$$J(t) = \int_{\mathbb{R}^d} g^2(\zeta; t) d\zeta = \int_{\mathbb{R}^q} \left(1 + \left| \sum_{i=1}^q \zeta_i - t \right| \right)^{-2\alpha-2K} \prod_{i=1}^q |\zeta_i|^{-2d} \prod_{i=1}^q d\zeta_i .$$



We now show that Lemma 8.4 applies with  $M_1 = 2\alpha + 2K$ ,  $M_2 = 0$  and  $\beta_i = 2d$  for  $i = 1, \dots, q$ . Indeed, we have  $M_2 - M_1 = -2\alpha - 2K \leq -2\alpha < -1$ . Further, for all  $\ell = 1, \dots, q - 1$ , we have, by the assumption on  $d$ ,

$$\sum_{i=\ell}^q \beta_i = 2d(1 + q - \ell) > (1 + q - \ell)(1 - 1/q) = q - \ell + (\ell - 1)/q \geq q - \ell.$$

Finally, since  $\alpha > 1/2$ , one has  $M_2 - M_1 + q = -2\alpha - 2K + q < q - 1 \leq \sum_i \beta_i$ .

Applying Lemma 8.4, we get  $J(t) \rightarrow 0$  as  $|t| \rightarrow \infty$  and  $J(0) < \infty$ . Thus, if  $s \neq 0$ , one has  $t = 2\pi\gamma_j s \rightarrow \infty$  as  $j \rightarrow \infty$  and hence we obtain (58). If  $s = 0$ , then  $t = 2\pi\gamma_j s = 0$  and using the bound (60),  $J(0) < \infty$ , and the dominated convergence theorem, we have that the convergence (57) follows from the convergence at a.e.  $\zeta \in \mathbb{R}^q$  of the left hand side of (60), which we now establish. Recall that  $f_{m,k}$  is defined in (54). By (6), (1) and the continuity of  $f^*$  at the origin, we have, as  $j \rightarrow \infty$ ,

$$\begin{aligned} \gamma_j^{-1/2} \widehat{h}_{j+m} \circ \Sigma_q(\zeta/\gamma_j) &= \left( \frac{\gamma_{j+m}}{\gamma_j} \right)^{1/2} \gamma_{j+m}^{-1/2} \widehat{h}_{j+m} \circ \Sigma_q((\zeta/\gamma_{j+m})(\gamma_{j+m}/\gamma_j)) \\ &\rightarrow \bar{\gamma}_m^{1/2} \widehat{h}_\infty(\bar{\gamma}_m(\zeta_1 + \dots + \zeta_q)), \end{aligned}$$

and for every  $\ell = 1, \dots, q$

$$\gamma_j^{-2d} f(\zeta_\ell/\gamma_j) = \gamma_j^{-2d} |1 - e^{-i\zeta_\ell/\gamma_j}|^{-2d} f^*(\zeta_\ell/\gamma_j) \rightarrow f^*(0) |\zeta_\ell|^{-2d}.$$

Hence  $\gamma_j^{-(d(q)+K)} f_{m,k}(\zeta; j, q, 0) \mathbb{1}_{\Gamma(q,s)}(\gamma_j^{-1}\zeta)$  converges to

$$(\bar{\gamma}_m)^{1/2} (f^*(0))^{q/2} \frac{e^{ik\bar{\gamma}_m(\zeta_1 + \dots + \zeta_q)} \times \widehat{h}_\infty(\bar{\gamma}_m(\zeta_1 + \dots + \zeta_q))}{(i(\zeta_1 + \dots + \zeta_q))^K |\zeta_1|^d \dots |\zeta_q|^d}.$$

This concludes the proof.  $\square$

## 6.2. Proof of Proposition 6.2

We now consider the short-range dependence part of the wavelet coefficients  $(W_{j,k})$  defined by (29) and (48). These wavelet coefficients can be equivalently defined as

$$W_{j,k}^{(SD)} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) \Delta^K Y_\ell^{(SD)}, \quad (61)$$

where we have set

$$\Delta^K Y_\ell^{(SD)} = \sum_{q \geq q_c + 1} \frac{c_q}{q!} H_q(X_\ell), \quad \ell \in \mathbb{Z}.$$

Using Lemma 4.1, since (26) holds and  $\{H_q(X_\ell)\}_{\ell \in \mathbb{Z}}$  are uncorrelated weakly stationary processes, the process  $\{\Delta^K Y_\ell^{(SD)}\}_{\ell \in \mathbb{Z}}$  is weakly stationary with spectral density

$$f^{(SD)}(\lambda) = \sum_{q \geq q_c + 1} \frac{c_q^2}{q!} f^{(*q)}(\lambda), \quad \lambda \in (-\pi, \pi).$$

By Lemma 4.1(ii), we have that  $\|f^{(\star\{q_c+1\})}\|_\infty < \infty$ . Using that  $\|g_1 \star g_2\|_\infty \leq \|g_1\|_\infty \|g_2\|_1$  and  $\|f\|_1 = 1$  by assumption, an induction yields

$$\sup_{q>q_c} \|f^{(\star q)}\|_\infty \leq \|f^{(\star\{q_c+1\})}\|_\infty .$$

Hence, by (26), we get  $\|f^{(SD)}\|_\infty < \infty$ . It follows that, for  $W_{j,k}^{(SD)}$  defined in (61), there is a positive constant  $C$  such that,

$$\mathbb{E}[W_{j,k}^{(SD)2}] \leq \|f^{(SD)}\|_\infty \int_{-\pi}^{\pi} |\widehat{h}_j^{(K)}(\lambda)|^2 d\lambda \leq C \int_0^\pi |\lambda|^{-2K} |\widehat{h}_j(\lambda)|^2 d\lambda = O(\gamma_j^{2K}) ,$$

where we used (4) with  $M \geq K$  and  $\alpha > 1/2$ . This last relation implies (51) and concludes the proof of Proposition 6.2.  $\square$

## 7. Proof of Lemmas 5.1 and 5.2

### 7.1. Proof of Lemma 5.1

Let us first prove that if  $\theta \in \mathcal{S}_{q,d}^{(K)}$  then  $Z_{q,d}^{(K)}(\theta)$  exists. Indeed, by Definition 5.1,  $Z_{q,d}^{(K)}(\theta)$  exists if

$$\int_{\mathbb{R}^q} \frac{|\widehat{\theta}(u_1 + \dots + u_q)|^2}{|u_1 + \dots + u_q|^{2K} |u_1 \dots u_q|^{2d}} du_1 \dots du_q < \infty . \quad (62)$$

Use now Lemma 8.3 with  $\beta_1 = \dots = \beta_q = -2d$  and  $f(x) = |\widehat{\theta}(x)|^2/|x|^{2K}$  and deduce that Condition (62) is equivalent to

$$\Gamma \int_{\mathbb{R}} |\widehat{\theta}(s)|^2 |s|^{q-1-2qd-2K} ds < \infty , \quad (63)$$

where

$$\Gamma = \prod_{i=2}^q \left( \int_{\mathbb{R}} |t|^{q-i-2d(q-i+1)} |1-t|^{-2d} dt \right) .$$

Note that the conditions  $0 < d < 1/2$  and  $0 < q < 1/(1-2d)$  ensure that  $\Gamma$  is finite. Further, Relation (63) implies  $\theta \in \mathcal{S}_{q,d}^{(K)}$ .

We now prove that for any  $m, k$ ,  $h_{\infty,m,k} \in \mathcal{S}_{q,d}^{(K)}$  when  $K \in \{0, \dots, M\}$ . By Definition (40) of  $h_{\infty,m,k}$

$$\widehat{h}_{\infty,m,k}(\xi) = (\overline{\gamma}_m)^{1/2} e^{-i\overline{\gamma}_m \xi} \overline{\widehat{h}_{\infty}(\overline{\gamma}_m \xi)} .$$

Hence

$$\int_{\mathbb{R}} |\widehat{h}_{\infty,m,k}(s)|^2 |s|^{q-1-2qd-2K} ds = \overline{\gamma}_m \int_{\mathbb{R}} |\widehat{h}_{\infty}(\overline{\gamma}_m s)|^2 |s|^{q-1-2qd-2K} ds .$$

Set  $v = \overline{\gamma}_m s$  and deduce that  $h_{\infty,m,k} \in \mathcal{S}_{q,d}^{(K)}$  is equivalent to

$$\overline{\gamma}_m^{2-(q-1-2qd-2K)} \int_{\mathbb{R}} |\widehat{h}_{\infty}(v)|^2 |v|^{q-1-2qd-2K} dv < \infty .$$

Assumption (12) implies that

$$\int_{\mathbb{R}} |\widehat{h}_{\infty}(v)|^2 |v|^{q-1-2qd-2K} dv \leq \int_{\mathbb{R}} \frac{|v|^{2M}}{(1+|v|)^{2M+2\alpha}} |v|^{q-1-2qd-2K} dv .$$

Since  $M \geq K$  and  $q(1-2d) \in (0, 1)$  then  $2M+q-1-2qd-2K = (2M-2K)+q(1-2d)-1 > -1$ . Further  $\alpha > 1/2$  and  $q(1-2d) \in (0, 1)$  imply that  $2M-2K-2\alpha+(q-1-2qd-2K) = -2\alpha-2K+q(1-2d)-1 < -1$ . Then

$$\int_{\mathbb{R}} |\widehat{h}_{\infty,m,k}(s)|^2 |s|^{q-1-2qd-2K} ds < \infty .$$

holds and  $h_{\infty,m,k} \in \mathcal{S}_{q,d}^{(K)}$ .

### 7.2. Proof of Lemma 5.2

Let  $a_t(u_1, \dots, u_q)$  denote the kernel of the integral in (42) defining  $\tilde{Z}_{q,d}^{(K)}$  and suppose we can exchange the order of integration and write

$$\int_{\mathbb{R}} \tilde{Z}_{q,d}^{(K)}(t) \theta(t) dt = \int_{\mathbb{R}^q} \left[ \int_{\mathbb{R}} a_t(u_1, \dots, u_q) \theta(t) dt \right] d\widehat{W}(u_1) \cdots d\widehat{W}(u_q) . \quad (64)$$

Then condition (43) gives

$$\int_{\mathbb{R}} \left[ e^{it(u_1+\dots+u_q)} - \sum_{\ell=0}^{K-1} \frac{(it(u_1+\dots+u_q))^{\ell}}{\ell!} \right] \overline{\theta(t)} dt = \int_{\mathbb{R}} e^{it(u_1+\dots+u_q)} \overline{\theta(t)} dt = \overline{\widehat{\theta} \circ \Sigma_q(u)} ,$$

showing that (64) equals  $\tilde{Z}_{q,d}^{(K)}(\theta)$  defined in (39). It remains to justify the change of order of integration in (64) by using a stochastic Fubini theorem, (see for instance [20, Theorem 2.1]). A sufficient condition is

$$\int_{\mathbb{R}} (a_t^2(u_1, \dots, u_q) du_1 \cdots du_q)^{1/2} dt < \infty .$$

This condition is satisfied, because setting  $v = tu$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^q} \left| e^{it(u_1+\dots+u_q)} - \sum_{\ell=0}^{K-1} \frac{(it(u_1+\dots+u_q))^{\ell}}{\ell!} \right|^2 |i(u_1+\dots+u_q)|^{-2K} |u_1 \cdots u_q|^{-2d} d^q u , \\ & \leq |t|^{2K+2d-q} \int_{\mathbb{R}^q} (1+|u_1+\dots+u_q|)^{-2K} |u_1 \cdots u_q|^{-2d} d^q u . \end{aligned}$$

## 8. Auxiliary lemmas

The following lemma provides a bound for the convolution of two functions exploding at the origin and decaying polynomially at infinity.

**Lemma 8.1.** *Let  $\alpha > 1$  and  $\beta_1, \beta_2 \in [0, 1)$  such that  $\beta_1 + \beta_2 < 1$ , and set*

$$g_i(t) = |t|^{-\beta_i}(1 + |t|)^{\beta_i - \alpha}.$$

Then

$$\sup_{u \in \mathbb{R}} \left( (1 + |u|)^\alpha \int_{\mathbb{R}} g_1(u - t) g_2(t) dt \right) < \infty. \quad (65)$$

**Proof.** We first show that

$$J(u) = \int_{\mathbb{R}} g_1(u - t) g_2(t) dt = \int_{\mathbb{R}} |u - t|^{-\beta_1} (1 + |u - t|)^{\beta_1 - \alpha} |t|^{-\beta_2} (1 + |t|)^{\beta_2 - \alpha} dt$$

is uniformly bounded on  $\mathbb{R}$ . Using the assumptions on  $\beta_1, \beta_2$ , there exist  $p > 1$  such that  $\beta_1 < 1/p < 1 - \beta_2$ . Let  $q$  be such that  $1/p + 1/q = 1$ . The Hölder inequality implies that

$$J(u)^{pq} \leq \int_{\mathbb{R}} |t|^{-p\beta_1} (1 + |t|)^{p\beta_1 - p\alpha} dt \times \int_{\mathbb{R}} |t|^{-q\beta_2} (1 + |t|)^{q\beta_2 - q\alpha} dt.$$

The condition on  $\alpha, \beta_1, \beta_2, p$  and the definition of  $q$  imply that these two integrals are finite. Hence  $\sup_u J(u) < \infty$ .

We now determine how fast  $J(u)$  tends to 0 as  $u \rightarrow \infty$ . Observe that, if  $|t - u| \leq |u|/2$ , then  $|t| \geq |u|/2$ . By splitting the integral in two integrals on the domains  $|t - u| \leq |u|/2$  and  $|t - u| > |u|/2$ , we get  $J(u) \leq J_1(u) + J_2(u)$  with

$$J_1(u) \leq (|u|/2)^{-\beta_2} (1 + |u|/2)^{\beta_2 - \alpha} \int_{\mathbb{R}} |u - t|^{-\beta_1} (1 + |t - u|)^{\beta_1 - \alpha} dt,$$

and

$$J_2(u) \leq (|u|/2)^{-\beta_1} (1 + |u|/2)^{\beta_1 - \alpha} \int_{\mathbb{R}} |t|^{-\beta_2} (1 + |t|)^{\beta_2 - \alpha} dt.$$

Now, as  $|u| \rightarrow \infty$ , we have  $J_i(u) = O(|u|^{-\alpha})$  for  $i = 1, 2$ , which achieves the proof.  $\square$

The next lemma describes the convolutions of two periodic functions that explode at the origin as a power. A different definition of convolution is involved here (see (33)).

**Lemma 8.2.** *Let  $(\beta_1, \beta_2) \in (0, 1)^2$ . Let  $g_1, g_2$  be  $(2\pi)$ -periodic functions such that  $g_i(\lambda) = |\lambda|^{-\beta_i} g_i^*(\lambda)$ ,  $i = 1, 2$ . Each  $g_i^*(\lambda)$  is a  $(2\pi)$ -periodic non-negative function, bounded on  $(-\pi, \pi)$  and positive at the origin, where it is also continuous. Let  $g = g_1 \star g_2$  as defined in (33). Then,*

- If  $\beta_1 + \beta_2 < 1$ ,  $g$  is bounded and continuous on  $(-\pi, \pi)$ , and satisfies  $g(0) > 0$ .

- If  $\beta_1 + \beta_2 > 1$ ,

$$g(\lambda) = |\lambda|^{-(\beta_1 + \beta_2 - 1)} g^*(\lambda),$$

where  $g^*(\lambda)$  is bounded on  $(-\pi, \pi)$  and converges to a positive constant as  $\lambda \rightarrow 0$ . If moreover for some  $\beta \in (0, 2]$  such that  $\beta < \beta_1 + \beta_2 - 1$  and some  $L > 0$ , one has for any  $i \in \{1, 2\}$

$$|g_i^*(\lambda) - g_i^*(0)| \leq L|\lambda|^\beta, \forall \lambda \in (-\pi, \pi), \quad (66)$$

then there exists some  $L' > 0$  depending only on  $L, \beta_1, \beta_2$  such that

$$|g^*(\lambda) - g^*(0)| \leq L'|\lambda|^\beta, \forall \lambda \in (-\pi, \pi).$$

**Proof.** By (33) and  $(2\pi)$ -periodicity, we may write

$$g(\lambda) = \int_{-\pi}^{\pi} g_1(u)g_2(\lambda - u) du = \int_{-\pi}^{\pi} |\{\lambda - u\}|^{-\beta_1} g_1^*(\lambda - u) |u|^{-\beta_2} g_2^*(u) du. \quad (67)$$

Let us first consider the case  $\beta_1 + \beta_2 < 1$ . We clearly have  $g(0) > 0$ . To prove that  $g$  is bounded, we proceed as in the case of convolutions of non-periodic functions (see the proof of Lemma 8.1), namely, for  $p, q$  such that  $\beta_1 < 1/p < 1 - \beta_2$  and  $1/p + 1/q = 1$ , the Hölder inequality gives that

$$\|g\|_\infty^{pq} \leq \|g_1\|_p^p \|g_2\|_q^q \leq \|g_1^*\|_\infty^p \|g_2^*\|_\infty^q \int_{-\pi}^{\pi} |t|^{-p\beta_1} dt \times \int_{-\pi}^{\pi} |t|^{-q\beta_2} dt < \infty. \quad (68)$$

For any  $\epsilon > 0$  and  $i = 1, 2$ , let  $g_{\epsilon,i}$  be the  $(2\pi)$ -periodic function such that for all  $\lambda \in (-\pi, \pi)$ ,  $g_{\epsilon,i}(\lambda) = \mathbb{1}_{(-\epsilon, \epsilon)}(\lambda) g_i(\lambda)$  and let  $\bar{g}_{\epsilon,i} = g_i - g_{\epsilon,i}$ . Then  $g = \bar{g}_{\epsilon,1} \star \bar{g}_{\epsilon,2} + g_{\epsilon,1} \star \bar{g}_{\epsilon,2} + \bar{g}_{\epsilon,1} \star g_{\epsilon,2} + g_{\epsilon,1} \star g_{\epsilon,2}$ . Since  $\bar{g}_{\epsilon,i}$  is bounded for  $i = 1, 2$ , we have that  $\bar{g}_{\epsilon,1} \star \bar{g}_{\epsilon,2}$  is continuous. On the other hand, using the Hölder inequality as in (68), we get that  $\|g_{\epsilon,1} \star \bar{g}_{\epsilon,2}\|_\infty, \|\bar{g}_{\epsilon,1} \star g_{\epsilon,2}\|_\infty, \|\bar{g}_{\epsilon,1} \star \bar{g}_{\epsilon,2}\|_\infty$  tend to zero as  $\epsilon \rightarrow 0$ . Hence  $g$  is continuous as well.

We now consider the case  $\beta_1 + \beta_2 \geq 1$ . Setting  $v = u/\lambda$  in (67), we get, for any  $\lambda \in [-\pi, \pi] \setminus \{0\}$ ,

$$g^*(\lambda) = |\lambda|^{\beta_1 + \beta_2 - 1} g(\lambda) = \int_{\mathbb{R}} \mathbb{1}_{(-\pi/|\lambda|, \pi/|\lambda|)}(v) |\{(1-v)\}_\lambda|^{-\beta_1} |v|^{-\beta_2} g_1^*(\lambda(1-v)) g_2^*(\lambda v) dv,$$

where for any real number  $x$  and  $\lambda \neq 0$ ,  $\{x\}_\lambda$  denotes the unique element of  $[-\pi/|\lambda|, \pi/|\lambda|]$  such that  $x - \{x\}_\lambda \in \mathbb{Z}$ . Take now  $|\lambda|$  small enough so that  $\pi/|\lambda| > 2$ . Then, for any  $v \in (-\pi/|\lambda| + 1, \pi/|\lambda|]$ , we have  $|\{(1-v)\}_\lambda| = |1-v| \geq |1-|v||$  and, for any  $v \in (-\pi/|\lambda|, -\pi/|\lambda| + 1]$ , we have

$$|\{(1-v)\}_\lambda| = |1-v-2\pi/|\lambda|| = 2\pi/|\lambda| + v - 1 \geq -v - 1 = |1-|v||. \quad (69)$$

Thus we have  $\mathbb{1}_{(-\pi/|\lambda|, \pi/|\lambda|)}(v) |\{(1-v)\}_\lambda|^{-\beta_1} \leq |1-|v||^{-\beta_1}$  for all  $v \in \mathbb{R}$ . We conclude that for  $|\lambda|$  small enough, the integrand in the last display is bounded from above by

$|1 - |v||^{-\beta_1} |v|^{-\beta_2} \|g_1^*\|_\infty \|g_2^*\|_\infty$ , which is integrable on  $v \in \mathbb{R}$ . Hence  $g^*$  is bounded, and by dominated convergence, as  $\lambda \rightarrow 0$ ,

$$g^*(\lambda) \rightarrow g_1^*(0)g_2^*(0) \int_{\mathbb{R}} |1 - v|^{-\beta_1} |v|^{-\beta_2} dv > 0. \quad (70)$$

We set  $g^*(0)$  equal to this limit.

Suppose moreover that  $g_1^*, g_2^*$  satisfy (66). We take  $g_1^*(0) = g_2^*(0) = 1$  without loss of generality and denote  $r_i(\lambda) = |g_i^*(\lambda) - 1|$  for  $i = 1, 2$ . Then  $r(\lambda) = |g^*(\lambda) - g^*(0)|$ , where  $g^*(0)$  is defined as the limit in (70), is at most

$$\int_{\mathbb{R}} |\mathbb{1}_{(-\pi/|\lambda|, \pi/|\lambda|)}(v) |\{(1 - v)\}_\lambda|^{-\beta_1} |v|^{-\beta_2} g_1^*(\lambda(1 - v))g_2^*(\lambda v) - |1 - v|^{-\beta_1} |v|^{-\beta_2}| dv.$$

Setting  $g_i^*(\lambda) = (g_i^*(\lambda) - 1) + 1$ , we have  $r \leq A + B_1 + B_2 + C$  with

$$\begin{aligned} A(\lambda) &= \int_{\mathbb{R}} |\mathbb{1}_{(-\pi/|\lambda|, \pi/|\lambda|)}(v) |\{(1 - v)\}_\lambda|^{-\beta_1} |v|^{-\beta_2} - |1 - v|^{-\beta_1} |v|^{-\beta_2}| dv, \\ B_i(\lambda) &= \int_{\mathbb{R}} \mathbb{1}_{(-\pi/|\lambda|, \pi/|\lambda|)}(v) |\{(1 - v)\}_\lambda|^{-\beta_j} |v|^{-\beta_i} r_i(\lambda v) dv, \end{aligned}$$

where  $(i, j)$  is  $(1, 2)$  or  $(2, 1)$ , and

$$C(\lambda) = \int_{\mathbb{R}} \mathbb{1}_{(-\pi/|\lambda|, \pi/|\lambda|)}(v) |\{(1 - v)\}_\lambda|^{-\beta_1} |v|^{-\beta_2} r_1(\lambda(1 - v))r_2(\lambda v) dv.$$

Since  $\{(1 - v)\}_\lambda = 1 - v$  for  $v \in [-\pi/|\lambda| + 1, \pi/|\lambda|)$  and  $\lambda$  large enough, we have

$$\begin{aligned} A(\lambda) &= \int_{(-\pi/|\lambda|, \pi/|\lambda|)^c} |1 - v|^{-\beta_1} |v|^{-\beta_2} dv \\ &\quad + \int_{-\pi/|\lambda|}^{-\pi/|\lambda|+1} \left| |\{(1 - v)\}_\lambda|^{-\beta_1} |v|^{-\beta_2} - |1 - v|^{-\beta_1} |v|^{-\beta_2} \right| dv. \end{aligned}$$

The first integral is  $O(|\lambda|^{\beta_1 + \beta_2 - 1})$ . Using (69), the second line of the last display is less than

$$\int_{\pi/|\lambda|-1}^{\pi/|\lambda|} [ |1 - v|^{-\beta_1} |v|^{-\beta_2} + |1 + v|^{-\beta_1} v^{-\beta_2} ] dv = O(|\lambda|^{\beta_1 + \beta_2}).$$

We conclude that as  $\lambda \rightarrow 0$ ,  $A(\lambda) = O(|\lambda|^{\beta_1 + \beta_2 - 1})$ . Moreover using that  $r_i(\lambda) \leq L|\lambda|^\beta$  and  $\beta_1 + \beta_2 - \beta > 1$ , we have  $B_i(\lambda) = O(|\lambda|^\beta)$  for  $i = 1, 2$ . The same is true for  $C$  since  $r_1$  and  $r_2$  are also bounded on  $\mathbb{R}$ . This achieves the proof.  $\square$

**Lemma 8.3.** *Let  $p$  be a positive integer and  $f : \mathbb{R} \rightarrow \mathbb{R}_+$ . Then, for any  $\beta \in \mathbb{R}^q$ ,*

$$\int_{\mathbb{R}^q} f(y_1 + \dots + y_q) \prod_{i=1}^q |y_i|^{\beta_i} dy_1 \dots dy_q = \Gamma \times \int_{\mathbb{R}} f(s) |s|^{q-1 + \beta_1 + \dots + \beta_q} ds, \quad (71)$$

where, for all  $i \in \{1, \dots, q\}$ ,  $B_i = \beta_i + \dots + \beta_q$  and

$$\Gamma = \prod_{i=2}^q \left( \int_{\mathbb{R}} |t|^{q-i+B_i} |1-t|^{\beta_{i-1}} dt \right).$$

(We note that  $\Gamma$  may be infinite in which case (71) holds with the convention  $\infty \times 0 = 0$ ).

**Proof.** Relation (71) is obtained by using the following two successive change of variables followed by an application of the Fubini Theorem. Setting, for all  $i = 1, \dots, q$ ,  $u_i = \sum_{j=i}^q y_j$ , we get that  $y_i = u_i - u_{i+1}$  for  $i < q$  and  $y_q = u_q$ . Then the integral in the left-hand side of (71) reads

$$\int_{\mathbb{R}^q} f(u_1) \left[ |u_q|^{\beta_q} \prod_{i=1}^{q-1} |u_i - u_{i+1}|^{\beta_i} \right] du_1 \cdots du_q. \quad (72)$$

The second change of variables consists in setting, for all  $i = 1, \dots, q$ ,  $u_i = \prod_{j=1}^i t_j$ . Then

$$du_1 \cdots du_q = \left( \prod_{i=1}^{q-1} t_i^{q-i} \right) dt_1 \cdots dt_q,$$

$$\prod_{i=1}^{q-1} |u_i - u_{i+1}|^{\beta_i} = \prod_{i=1}^{q-1} (|t_1 \cdots t_i|^{\beta_i} |1-t_{i+1}|^{\beta_i}) = \left( \prod_{i=1}^{q-1} |t_i|^{\beta_i + \dots + \beta_{q-1}} \right) \left( \prod_{i=2}^q |1-t_i|^{\beta_{i-1}} \right),$$

and  $|u_q| = \prod_{i=1}^q |t_i|^{\beta_q}$ , so that (72) becomes

$$\int_{\mathbb{R}^q} f(t_1) \prod_{i=1}^q |t_i|^{\beta_i + \dots + \beta_{q+q-i}} \prod_{i=2}^q |1-t_i|^{\beta_{i-1}} dt_1 \cdots dt_q,$$

which by Fubini Theorem yields the required result. □

**Lemma 8.4.** *Let  $a \in \mathbb{R}$  and  $q$  be a positive integer. Let  $\beta = (\beta_1, \dots, \beta_q) \in (-\infty, 1)^q$ ,  $M_1 > 0$  and  $M_2 > -1$  such that  $M_2 - M_1 < -1$ . Assume that  $q + M_2 - M_1 < \sum_{i=1}^q \beta_i$ , and that for any  $\ell \in \{1, \dots, q-1\}$ ,  $\sum_{i=\ell}^q \beta_i > q - \ell$ . Set for any  $a \in \mathbb{R}$ ,*

$$J_q(a; M_1, M_2; \beta) = \int_{\mathbb{R}^q} \frac{|\sum_q(\zeta) - a|^{M_2}}{(1 + |\sum_q(\zeta) - a|)^{M_1} \prod_{i=1}^q |\zeta_i|^{\beta_i}} d\zeta.$$

Then one has

$$\sup_{a \in \mathbb{R}} (1 + |a|)^{1-q + \sum_{i=1}^q \beta_i} J_q(a; M_1, M_2; \beta) < \infty. \quad (73)$$

In particular,

$$J_q(0; M_1, M_2; \beta) < \infty,$$

and

$$J_q(a; M_1, M_2; \beta) = O(|a|^{-(1-q+\sum_{i=1}^q \beta_i)}) \quad \text{as } a \rightarrow \infty.$$

**Proof.** Since  $J_q(a; M_1, M_2; \beta_1, \dots, \beta_q) = J_q(-a; M_1, M_2; \beta)$ , we may suppose  $a \geq 0$ . By Lemma 8.3,

$$J_q(a; M_1, M_2; \beta_1, \dots, \beta_q) = \Gamma \int_{\mathbb{R}} \frac{|s-a|^{M_2} |s|^{q-1-(\beta_1+\dots+\beta_q)}}{(1+|s-a|)^{M_1}} ds$$

where

$$\Gamma = \prod_{i=2}^q \int_{\mathbb{R}} \frac{dt}{|t|^{\beta_i+\dots+\beta_q-(q-i)} |1-t|^{\beta_{i-1}}}.$$

The conditions on  $\beta_i$ 's,  $M_1$  and  $M_2$  imply  $J_q(a; M_1, M_2; \beta_1, \dots, \beta_q) < \infty$  for all  $a$ . To obtain the sup on  $a > 0$ , we set  $v = s/a$ . Then, denoting  $S = \sum_{i=1}^q \beta_i$ , we get

$$J_q(a; M_1, M_2; \beta) = C a^{q+M_2-S} \int_{\mathbb{R}} |v-1|^{M_2} (1+a|v-1|)^{-M_1} |v|^{-S+(q-1)} dv, \quad (74)$$

where  $C$  is a positive constant. We separate the integration domain in two. Suppose first that  $|v-1| \leq a^{-1}$ . Then in this case we have  $(1+a|v-1|)^{-M_1} \leq 1$ . Since  $|v|$  is bounded on the interval  $|v-1| < a^{-1}$  for  $a$  large then as  $a \rightarrow \infty$ ,

$$\int_{|v-1| \leq a^{-1}} |v-1|^{M_2} (1+a|v-1|)^{-M_1} |v|^{-S+(q-1)} dv = O\left(\int_{|v-1| \leq a^{-1}} |v-1|^{M_2} dv\right) = O(a^{-1-M_2}).$$

Now suppose that  $|v-1| > a^{-1}$ . Then  $(1+a|v-1|)^{-M_1} \leq (a|v-1|)^{-M_1}$ , and

$$\begin{aligned} I &= \int_{|v-1| > a^{-1}} |v-1|^{M_2} (1+a|v-1|)^{-M_1} |v|^{-S+(q-1)} dv \\ &\leq a^{-M_1} \int_{|v-1| > a^{-1}} |v-1|^{M_2-M_1} |v|^{-S+(q-1)} dv \\ &= a^{-M_1} \left( \int_{|v| \geq 2} |v-1|^{M_2-M_1} |v|^{-S+(q-1)} dv + \int_{1/2 \leq |v| \leq 2, |v-1| > a^{-1}} |v-1|^{M_2-M_1} |v|^{-S+(q-1)} dv \right) \\ &\quad + a^{-M_1} \int_{|v| \leq 1/2, |v-1| > a^{-1}} |v-1|^{M_2-M_1} |v|^{-S+(q-1)} dv. \end{aligned}$$

The first integral concentrates around  $v = \infty$ , the second around  $v = 1$  and the third around  $v = 0$ . The first integral is bounded, the second is

$$O\left(\int_{|v-1| > a^{-1}} |v-1|^{M_2-M_1} dv\right) = O(a^{M_1-M_2-1}), \quad \text{as } a \rightarrow \infty,$$

and the third is bounded. Therefore we get

$$I = O(a^{-M_1}) + O(a^{-M_2-1}),$$



since  $M_2 - M_1 < -1$ . Thus (74) gives

$$J_q(a; M_1, M_2; \beta) = O(a^{-1+q-S}) \quad \text{as } a \rightarrow \infty,$$

yielding the bound (73). □

**Acknowledgements.** Murad S. Taqqu was supported in part by the NSF grants DMS-0608669 and DMS-1007616 at Boston University.

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