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Abstract

We study the asymptotic behavior of wavelet coefficients of random processes with long memory. These processes may be stationary or not and are obtained as the output of non-linear filter with Gaussian input. The wavelet coefficients that appear in the limit are random, typically non-Gaussian and belong to a Wiener chaos. They can be interpreted as wavelet coefficients of a generalized self-similar process.

Keywords: Hermite processes, Wavelet coefficients, Wiener chaos, self-similar processes, Long-range dependence.

2000 MSC: Primary 42C40, 60G18, 62M15, Secondary: 60G20, 60G22

1. Introduction

Let $X = \{X_n\}_{n \in \mathbb{Z}}$ be a stationary Gaussian process with mean zero, unit variance and spectral density $f(\lambda), \lambda \in (-\pi, \pi]$ and thus covariance equal to

$$r(n) = \mathbb{E}(X_0X_n) = \int_{-\pi}^{\pi} e^{in\lambda} f(\lambda) d\lambda.$$  

The process $\{X_n\}_{n \in \mathbb{Z}}$ is said to have short memory or short-range dependence if $f(\lambda)$ is bounded around $\lambda = 0$ and long memory or long-range dependence if $f(\lambda) \to \infty$ as $\lambda \to 0$. We will suppose that $\{X_n\}_{n \in \mathbb{Z}}$ has long-memory with memory parameter $d > 0$, that is,

$$f(\lambda) \sim |\lambda|^{-2d} f^*(\lambda) \text{ as } \lambda \to 0$$

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where \( f^*(\lambda) \) is a bounded spectral density which is continuous and positive at the origin. It is convenient to interpret this behavior as the result of a fractional integrating operation, whose transfer function reads \( \lambda \mapsto (1 - e^{-i\lambda})^{-d} \). Hence we set
\[
f(\lambda) = |1 - e^{-i\lambda}|^{-2d} f^*(\lambda), \quad \lambda \in (-\pi, \pi].
\] (1)

We relax the above assumptions in two ways:

1. Consider, instead of the Gaussian process \( \{X_n\}_{n \in \mathbb{Z}} \) the non–Gaussian process \( \{G(X_n)\}_{n \in \mathbb{Z}} \) where \( G \) is a non–linear filter such that \( \mathbb{E}[G(X_n)] = 0 \) and \( \mathbb{E}[G(X_n)^2] < \infty \). The non–linear process \( \{G(X_n)\}_{n \in \mathbb{Z}} \) is said to be subordinated to the Gaussian process \( \{X_n\}_{n \in \mathbb{Z}} \).

2. Drop the stationarity assumption by considering a process \( \{Y_n\}_{n \in \mathbb{Z}} \) which becomes stationary when differenced \( K \geq 0 \) times.

We shall thus consider \( \{Y_n\}_{n \in \mathbb{Z}} \) such that
\[
(\Delta^K Y)_n = G(X_n), \quad n \in \mathbb{Z},
\]
where \( (\Delta Y)_n = Y_n - Y_{n-1} \) and where \( \{X_n\}_{n \in \mathbb{Z}} \) is Gaussian with spectral density \( f \) satisfying (1).

Since \( Y = \{Y_n\}_{n \in \mathbb{Z}} \) is random so will be its wavelet coefficients \( \{W_{j,k}, j \geq 0, k \in \mathbb{Z}\} \) which are defined below. Our goal is to find the distribution of the wavelet coefficients at large scales \( j \to \infty \). This is an important step in developing methods for estimating the underlying long memory parameter \( d \). The large scale behavior of the wavelet coefficients was studied in [1] in the case where there was no filter \( G \), that is, when \( Y \) is a Gaussian process such that \( \Delta^K Y = X \), and also in the case where \( Y \) is a non–Gaussian linear process (see [2]).

We obtain our random wavelet coefficients by using more general linear filters that those related to multiresolution analysis (MRA) (see for e.g. [3], [4]). In practice, however, the methods are best implemented using Mallat’s algorithm and a MRA. Our filters are denoted \( h_j \) where \( j \) is the scale and we use a scaling factor \( \gamma_j \uparrow \infty \) as \( j \uparrow \infty \). In the case of a MRA, \( \gamma_j = 2^j \) and \( h_j \) are generated by a (low pass) scaling filter and its corresponding quadratic (high pass) mirror filter. More generally one can use a scaling function \( \varphi \) and a mother wavelet \( \psi \) to generate the random wavelet coefficients by setting
\[
W_{j,k} = \int_{\mathbb{R}} \psi_{j,k}(t) \left( \sum_{\ell \in \mathbb{Z}} \varphi(t - \ell) Y_\ell \right) dt,
\] (2)
where \( \psi_{j,k} = 2^{-j/2} \psi(2^{-j}t - k), j \geq 0 \). Observe that we use here the engineering convention that large values of \( j \) correspond to large scales and hence low frequencies. If \( \varphi \) and \( \psi \) have compact support then the corresponding filters \( h_j \) have finite support of size \( O(2^j) \). For more details on related conditions on \( \varphi \) and \( \psi \) (see [1]).
The idea of using wavelets to estimate the long memory coefficient $d$ goes back to Wornell and al. ([5]) and Flandrin ([6, 7, 8, 9]). See also Abry and al. ([10, 11]). Those methods are an alternative to the Fourier methods developed by Fox and Taqqu ([12]) and Robinson ([13, 14]). For a general comparison of Fourier and wavelet approach, see [15]. The case of the Rosenblatt process, which is the Hermite process of order $q = 2$, was studied by [16].

The paper is structured as follows. In Section 2 we introduce the wavelet filters. The processes are defined in Section 3 using integral representations and Section 4 presents the so-called Wiener chaos decomposition. The main result and its interpretations is given in Section 5. It is proved in Section 6. Auxiliary lemmas are presented and proved in Sections 7 and 8.

2. Assumptions on the wavelet filter

The wavelet transform of $Y$ involves the application of a linear filter $h_j(\tau), \tau \in \mathbb{Z}$, at each scale $j \geq 0$. We shall characterize the filters $h_j$ by their discrete Fourier transform:

$$\hat{h}_j(\lambda) = \sum_{\tau \in \mathbb{Z}} h_j(\tau) e^{-i\lambda \tau}, \lambda \in [-\pi, \pi].$$

Assumptions on $\hat{h}_j$ are stated below. The resulting wavelet coefficients are defined as

$$W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j(\gamma_jk - \ell)Y_\ell, \hspace{1cm} j \geq 0, \hspace{0.2cm} k \in \mathbb{Z},$$

where $\gamma_j \uparrow \infty$ is a sequence of non-negative scale factors applied at scale $j$, for example $\gamma_j = 2^j$. We will assume that for any $m \in \mathbb{Z},$

$$\lim_{j \to \infty} \frac{\gamma_{j+m}}{\gamma_j} = \gamma_m > 0. \hspace{2cm} (3)$$

As noted, in this paper, we do not assume that the wavelet coefficients are orthogonal nor that they are generated by a multiresolution analysis. Our assumptions on the filters $h_j$ are as follows:

a. **Finite support**: For each $j$, $\{h_j(\tau)\}_{\tau \in \mathbb{Z}}$ has finite support.

b. **Uniform smoothness**: There exists $M \geq K$, $\alpha > 1/2$ and $C > 0$ such that for all $j \geq 0$ and $\lambda \in [-\pi, \pi],$

$$|\hat{h}_j(\lambda)| \leq \frac{C\gamma_j^{1/2}|\gamma_j\lambda|^M}{(1 + \gamma_j|\lambda|)^{M+\alpha}}. \hspace{2cm} (4)$$

By $2\pi$-periodicity of $\hat{h}_j$ this inequality can be extended to $\lambda \in \mathbb{R}$ as

$$|\hat{h}_j(\lambda)| \leq \frac{C\gamma_j^{1/2}|\gamma_j\{\lambda\}|^M}{(1 + \gamma_j|\{\lambda\}|)^{\alpha+M}}. \hspace{2cm} (5)$$

where $\{\lambda\}$ denotes the element of $(-\pi, \pi]$ such that $\lambda - \{\lambda\} \in 2\pi\mathbb{Z}$. 

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c. Asymptotic behavior: There exists some non identically zero function \( \hat{h}_\infty \) such that

\[
\lim_{j \to +\infty} (\gamma_j^{-1/2} \hat{h}_j(\gamma_j^{-1} \lambda)) = \hat{h}_\infty(\lambda).
\]

(6)

Observe that while \( \hat{h}_j \) is 2\( \pi \)-periodic, the function \( \hat{h}_\infty \) is a non-periodic function on \( \mathbb{R} \) (this follows from (12) below). For the connection between these assumptions on \( h_j \) and corresponding assumptions on the scaling function \( \varphi \) and the mother wavelet \( \psi \) in the classical wavelet setting (2) (see [1]). In particular, in that case, one has \( \hat{h}_\infty = \hat{\varphi}(0) \hat{\psi} \).

Our goal is to study the large scale behavior of the random wavelet coefficients

\[
W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j(\gamma_j k - \ell) Y_\ell = \sum_{\ell \in \mathbb{Z}} h_j(\gamma_j k - \ell) (\Delta^{-K} G(X))_\ell,
\]

(7)

where we set symbolically \( Y_\ell = (\Delta^{-K} G(X))_\ell \) for \( (\Delta^K Y)_\ell = G(X_\ell) \).

By Assumption (4), \( h_j \) has null moments up to order \( M - 1 \), that is, for any \( m \in \{0, \ldots, M - 1\} \),

\[
\sum_{\ell \in \mathbb{Z}} h_j(\ell) \ell^m = 0.
\]

(8)

Therefore, since \( M \geq K \), \( \hat{h}_j \) can be expressed as

\[
\hat{h}_j(\lambda) = (1 - e^{-i\lambda})^K \hat{h}_j^{(K)}(\lambda),
\]

(9)

where \( \hat{h}_j^{(K)} \) is also a trigonometric polynomial of the form

\[
\hat{h}_j^{(K)}(\lambda) = \sum_{\tau \in \mathbb{Z}} h_j^{(K)}(\tau) e^{-i\lambda \tau},
\]

(10)

since \( h_j^{(K)} \) has finite support for any \( j \). Then we obtain another way of expressing \( W_{j,k} \), namely,

\[
W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) G(X_\ell).
\]

(11)

We have thus incorporated the linear filter \( \Delta^{-K} \) in (7) into the filter \( h_j \) and denoted the new filter \( h_j^{(K)} \).

Remarks

1. Since \( \{G(X_\ell), \ell \in \mathbb{Z}\} \) is stationary, it follows from (11) that \( \{W_{j,k}, k \in \mathbb{Z}\} \) is stationary for each scale \( j \).

2. Observe that \( \Delta^K Y \) is centered by definition. However, by (8), the definition of \( W_{j,k} \) only depends on \( \Delta^M Y \). In particular, provided that \( M \geq K + 1 \), its value is not modified if a constant is added to \( \Delta^K Y \), whenever \( M \geq K + 1 \).
3. Assumptions (4) and (6) imply that for any \( \lambda \in \mathbb{R} \),

\[
|\hat{h}_\infty(\lambda)| \leq C \frac{|\lambda|^M}{(1 + |\lambda|)^{\alpha + M}}.
\]  

(12)

Hence \( \hat{h}_\infty \in L^2(\mathbb{R}) \) since \( \alpha > 1/2 \).

4. The Fourier transform of \( f \),

\[
\mathfrak{F}(f)(\xi) = \int_{\mathbb{R}^q} f(t) e^{-it\xi} \, dt, \quad \xi \in \mathbb{R}^q ,
\]  

(13)

is defined for any \( f \in L^2(\mathbb{R}^q, \mathbb{C}) \). We let \( h_\infty \) be the \( L^2(\mathbb{R}) \) function such that \( \hat{h}_\infty = \mathfrak{F}[h_\infty] \).

3. Integral representations

It is convenient to use an integral representation in the spectral domain to represent the random processes (see for example [17, 18]). The stationary Gaussian process \( \{X_k, k \in \mathbb{Z}\} \) with spectral density (11) can be written as

\[
X_\ell = \int_{-\pi}^{\pi} e^{i\lambda\ell} f^{1/2}(\lambda) d\hat{W}(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda\ell} f^{1/2}(\lambda) \frac{d\hat{W}(\lambda)}{|1 - e^{-i\lambda}|^d}, \quad \ell \in \mathbb{N} .
\]  

(14)

This is a special case of

\[
\hat{I}(g) = \int_{\mathbb{R}} g(x) d\hat{W}(x),
\]  

(15)

where \( \hat{W}(\cdot) \) is a complex–valued Gaussian random measure satisfying

\[
\mathbb{E}(\hat{W}(A)) = 0 \quad \text{for every Borel set } A \text{ in } \mathbb{R} ,
\]  

(16)

\[
\mathbb{E}(\hat{W}(A)\hat{W}(B)) = |A \cap B| \text{ for every Borel sets } A \text{ and } B \text{ in } \mathbb{R} ,
\]  

(17)

\[
\sum_{j=1}^{n} \hat{W}(A_j) = \hat{W}\left( \bigcup_{j=1}^{n} A_j \right) \text{ if } A_1, \cdots, A_n \text{ are disjoint Borel sets in } \mathbb{R} ,
\]  

(18)

\[
\hat{W}(A) = \overline{\hat{W}(-A)} \quad \text{for every Borel set } A \text{ in } \mathbb{R} .
\]  

(19)

The integral (15) is defined for any function \( g \in L^2(\mathbb{R}) \) and one has the isometry

\[
\mathbb{E}(|\hat{I}(g)|^2) = \int_{\mathbb{R}} |g(x)|^2 \, dx .
\]

The integral \( \hat{I}(g) \), moreover, is real–valued if

\[
g(x) = \overline{g(-x)} .
\]
We shall also consider multiple Itô–Wiener integrals

\[ \hat{I}_q(g) = \int_{\mathbb{R}^q} g(\lambda_1, \ldots, \lambda_q) d\hat{W}(\lambda_1) \cdots d\hat{W}(\lambda_q) \]

where the double prime indicates that one does not integrate on hyperdiagonals \( \lambda_i = \pm \lambda_j, i \neq j \). The integrals \( \hat{I}_q(g) \) are handy because we will be able to expand our non–linear functions \( G(X_k) \) introduced in Section 1 in multiple integrals of this type.

These multiples integrals are defined as follows. Denote by \( \overline{L}^2(\mathbb{R}^q, \mathbb{C}) \) the space of complex valued functions defined on \( \mathbb{R}^q \) satisfying

\[ g(-x_1, \ldots, -x_q) = g(x_1, \ldots, x_q) \text{ for } (x_1, \ldots, x_q) \in \mathbb{R}^q, \]

\[ \|g\|^2_{\overline{L}^2} := \int_{\mathbb{R}^q} |g(x_1, \ldots, x_q)|^2 \, dx_1 \cdots dx_q < \infty. \]

Let \( \hat{L}^2(\mathbb{R}^q, \mathbb{C}) \) denote the set of functions in \( \overline{L}^2(\mathbb{R}^q, \mathbb{C}) \) that are symmetric in the sense that \( g = \tilde{g} \) where \( \tilde{g}(x_1, \ldots, x_q) = 1/q! \sum_{\sigma} g(x_{\sigma(1)}, \ldots, x_{\sigma(q)}) \), where the sum is over all permutations of \( \{1, \ldots, q\} \). One defines now the multiple integral with respect to the spectral measure \( \hat{W} \) by a density argument. For a step function of the form

\[ g = \sum_{j_\ell = \pm 1, \ldots, \pm N} c_{j_1, \ldots, j_n} 1_{\Delta_{j_1}} \times \cdots \times 1_{\Delta_{j_n}} \]

where the \( c \)'s are real–valued, \( \Delta_{j_\ell} = -\Delta_{-j_\ell} \) and \( \Delta_{j_\ell} \cap \Delta_{j_m} = \emptyset \) if \( \ell \neq m \), one sets

\[ \hat{I}_q(g) = \sum_{j_\ell = \pm 1, \ldots, \pm N}'' c_{j_1, \ldots, j_n} \hat{W}(\Delta_{j_1}) \cdots \hat{W}(\Delta_{j_n}). \]

Here, \( \sum'' \) indicates that one does not sum over the hyperdiagonals, that is, when \( j_\ell = \pm j_m \) for \( \ell \neq m \). The integral \( \hat{I}_q \) verifies that

\[ \mathbb{E}(\hat{I}_q(g_1)\hat{I}_q'(g_2)) = \begin{cases} q! \langle g_1, g_2 \rangle_{L^2}, & \text{if } q = q' \\ 0, & \text{if } q \neq q'. \end{cases} \]

Observe, moreover, that for every step function \( g \) with \( q \) variables as above

\[ \hat{I}_q(g) = \hat{I}_q(\tilde{g}). \]

Since the set of step functions is dense in \( \overline{L}^2(\mathbb{R}^q, \mathbb{C}) \), one can extend \( \hat{I}_q \) to an isometry from \( \overline{L}^2(\mathbb{R}^q, \mathbb{C}) \) to \( L^2(\Omega) \) and the above properties hold true for this extension.

Remark. Property (20) of the function \( f \) in \( \overline{L}^2(\mathbb{R}^q, \mathbb{C}) \) together with Property (19) of \( \hat{W} \) ensure that \( \hat{I}_q(f) \) is a real–valued random variable.
4. Wiener Chaos

Our results are based on the expansion of the function $G$, introduced in Section 1, in Hermite polynomials. The Hermite polynomials are

$$H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q} \left( e^{-x^2/2} \right),$$

in particular, $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$. If $X$ is a normal random variable with mean 0 and variance 1, then

$$E(H_q(X)H_{q'}(X)) = \int_\mathbb{R} H_q(x)H_{q'}(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = q! \delta_{q,q'}.$$ 

Moreover,

$$G(X) = \sum_{q=1}^{+\infty} \frac{c_q}{q!} H_q(X),$$

where the convergence is in $L^2(\Omega)$ and where

$$c_q = E(G(X)H_q(X)).$$

The expansion (24) is called a Wiener chaos expansion with each term in the chaos expansion living in a different chaos. The expansion (24) starts at $q = 1$, since

$$c_0 = E(G(X)H_0(X)) = E(G(X)) = 0,$$

by assumption. The condition $E(G(X)^2) < \infty$ implies

$$\sum_{q=1}^{+\infty} \frac{c_q^2}{q!} < \infty.$$ 

Hermite polynomials are related to multiple integrals as follows: if $X = \int_\mathbb{R} g(x) d\hat{W}(x)$ with $E(X^2) = \int_\mathbb{R} |g(x)|^2 dx = 1$ and $g(x) = \overline{g(-x)}$ so that $X$ has unit variance and is real–valued, then

$$H_q(X) = \hat{I}_q(g^{\otimes q}) = \int_{\mathbb{R}^q} g(x_1) \cdots g(x_q) d\hat{W}(x_1) \cdots d\hat{W}(x_q).$$

The expansion (24) of $G$ induces a corresponding expansion of the wavelet coefficients $W_{j,k}$, namely,

$$W_{j,k} = \sum_{q=1}^{+\infty} \frac{c_q}{q!} W_{j,k}^{(q)}, (28)$$

where by (11) one has

$$W_{j,k}^{(q)} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) H_q(X_\ell).$$
The Gaussian sequence $\{X_n\}_{n \in \mathbb{Z}}$ is long-range dependent because its spectrum at low frequencies behaves like $|\lambda|^{-2d}$ with $d > 0$ and hence explodes at $\lambda = 0$. What about the processes $\{H_q(X_\ell)\}_\ell$ for $q \geq 2$? What is the behavior of the spectrum at low frequencies? Does it explodes at $\lambda = 0$? The answer depends on the respective values of $q$ and $d$. Let us define

$$q_c = \max\{q \in \mathbb{N} : q < 1/(1-2d)\}, \quad (30)$$

and

$$d(q) = qd + (1-q)/2. \quad (31)$$

One has

$$d(q) > 0 \quad \text{if} \quad q \leq q_c, \quad \text{that is if} \quad q < 1/(1-2d). \quad (32)$$

The following result shows that the spectral density of $\{H_q(X_\ell)\}_\ell \in \mathbb{Z}$ has a different behavior at zero frequency depending on whether $q \leq q_c$ or $q > q_c$. It is long-range dependent when $q \leq q_c$ and short-range dependent when $q > q_c$. We first give a definition.

**Definition 4.1.** The convolution of two locally integrable $(2\pi)$-periodic functions $g_1$ and $g_2$ is defined as

$$(g_1 \ast g_2)(\lambda) = \int_{-\pi}^{\pi} g_1(u)g_2(\lambda-u)du. \quad (33)$$

Moreover the $q$ times self-convolution of $g$ is denoted by $g^{(*q)}$.

**Lemma 4.1.** Let $q$ be a positive integer. The spectral density of $\{H_q(X_\ell)\}_\ell \in \mathbb{Z}$ is

$$q!f^{(*q)} = q!(f \ast \cdots \ast f),$$

where the spectral density $f$ of $\{X_\ell\}_\ell \in \mathbb{Z}$ is given in (1). Moreover the following holds:

(i) If $q \leq q_c$, then $\lambda^{2d(q)}f^{(*q)}(\lambda)$ is bounded on $\lambda \in (0,\pi)$ and converges to a positive number as $\lambda \downarrow 0$.

(ii) If $q > q_c$, then $f^{(*q)}(\lambda)$ is bounded on $\lambda \in (0,\pi)$ and converges to a positive number as $\lambda \downarrow 0$.

Hence if $q \leq q_c$, $\{H_q(X_\ell)\}_\ell$ has long memory with parameter $d(q) > 0$ whereas if $q > q_c$, $\{H_q(X)\}_\ell$ has a short-memory behavior.

**Proof.** By definition of $H_q$ and since $X$ has unit variance by assumption, we have

$$\mathbb{E}(H_q(X_\ell)H_q(X_{\ell+m})) = q! \left( \int_{-\pi}^{\pi} f(\lambda)e^{i\lambda m}d\lambda \right)^q.$$ 

Using the fact that, for any two locally integrable $(2\pi)$-periodic functions $g_1$ and $g_2$, one has

$$\int_{-\pi}^{\pi} (g_1 \ast g_2)(\lambda)e^{i\lambda m}d\lambda = \int_{-\pi}^{\pi} g_1(u)e^{ium}du \times \int_{-\pi}^{\pi} g_2(v)e^{ivm}dv,$$
we obtain that the spectral density of \(\{H_q(X_\ell)\}_\ell\) is \(q! f^{(*)q}\).

The properties of \(f^{(*)q}\) stated in Lemma 4.1 are proved by induction on \(q\) using Lemma 8.2. Observe indeed that if \(\beta_1 = d(q)\) and \(\beta_2 = 2d\), then
\[
\beta_1 + \beta_2 - 1 = 2d(q) + 2d - 1 = (2dq + 1 - q) + 2d - 1 = 2(q+1)d - (q+1) + 1 = 2d(q+1).
\]

Now, consider the expansion of \(\Delta^K Y_\ell = G(X_\ell) = \sum_{q=0}^{\infty} (c_q/q!) H_q(X_\ell)\), where
\[
q_0 = \min\{q \geq 1, c_q \neq 0\}.
\] (34)

The exponent \(q_0\) is called the Hermite rank of \(\Delta^K Y_\ell\).

In the following, we always assume that at least one summand of \(\Delta^K Y_\ell\) has long memory, that is, in view of Lemma 4.1,
\[
q_0 \leq q_c.\] (35)

5. The result and its interpretations

In this section we describe the limit in distribution of the wavelet coefficients \(\{W_{j+m,k}\}_{m,k}\) as \(j \to \infty\), adequately normalized, and we interpret the limit. Recall that \(W_{j+m,k}\) involves a sum of chaoses of all order. In the limit, however, only the order \(q_0\) will prevail. The convergence of finite–dimensional distributions is denoted by \(\triangleright_{\text{fdi}}\).

**Theorem 5.1.** As \(j \to \infty\), we have
\[
\left\{\gamma_j^{-(d(q_0)+K)} W_{j+m,k}, m, k \in \mathbb{Z}\right\} \xrightarrow{\text{fdi}} c_{q_0} (f^*(0))^{q_0/2} \left\{Y_{m,k}^{(q_0,K)}\right\},
\] (36)

where for every positive integer \(q\),
\[
Y_{m,k}^{(q,K)} = (\pi_m)^{1/2} \int_{\mathbb{R}^q} e^{i\pi_m (\zeta_1 + \cdots + \zeta_q)} \hat{h}_\infty (\pi_m (\zeta_1 + \cdots + \zeta_q)) \frac{\hat{W}(\zeta_1) \cdots \hat{W}(\zeta_q)}{\zeta_1^d \cdots \zeta_q^d} d\zeta_1 \cdots d\zeta_q.\] (37)

This Theorem is proved in Section 6.

**Interpretation of the limit.**

The limit distribution can be interpreted as the wavelet coefficients of a generalized Hermite process defined below, based on the wavelet family
\[
\left\{h_{\infty,m,k}(t) = \pi_m^{-1/2} h_\infty(-\pi_m^1 t + k), m, k \in \mathbb{Z}\right\}.\] (38)

This wavelet family is the natural one to consider because the Fourier transform \(\hat{h}_\infty(\lambda)\) is the rescaled limit of the original \(\hat{h}_j(\lambda)\) as indicated in (39).

A generalized process is indexed not by time but by functions. The generalized Hermite processes for any order \(q\) in \(\{1, \ldots, q_c\}\) are defined as follows:
Definition 5.1. Let $0 < d < 1/2$ and let $q$ be a positive integer such that $0 < q < 1/(1-2d)$ and $K \geq 0$. Define the set of functions

$$S^{(K)}_{q,d} = \left\{ \theta : \int_{\mathbb{R}} |\hat{\theta}(\xi)|^2 |\xi|^{q-1-2dq-2K} \text{d}\xi < \infty \right\},$$

where $\hat{\theta} = \mathcal{F}[\theta]$. The generalized random process $Z^{(K)}_{q,d}$ is indexed by functions $\theta \in S^{(K)}_{q,d}$ and is defined as

$$Z^{(K)}_{q,d}(\theta) = \int_{\mathbb{R}^q} \frac{\overline{\theta(u_1 + \cdots + u_q)}}{(u_1 \cdots u_q)^d} \text{d}\hat{W}(u_1) \cdots \text{d}\hat{W}(u_q),$$

where $\hat{\theta} = \mathcal{F}[\theta]$ as defined in (13).

Now fix $(m, k) \in \mathbb{Z}^2$ and choose a function $h_{\infty,m,k}(t), t \in \mathbb{R}$ as in (38), so that

$$\mathcal{F}[h_{\infty,m,k}](\xi) = \mathcal{F}[h_{\infty}(-T_m t + k)](\xi) = (\gamma_m)^{1/2} e^{-i\pi \frac{|\xi|}{\gamma_m}} h_{\infty}(\gamma_m \xi).$$

Lemma 5.1. The conditions on $d$ and $q$ in Definition 5.1 ensures the existence of $Z^{(K)}_{q,d}(\theta)$. In particular,

$$h_{\infty,m,k} \in S^{(K)}_{q,d} \text{ for all } K \in \{0, \ldots, M\},$$

and hence $Z^{(K)}_{q,d}(h_{\infty,m,k})$ is well-defined.

This Lemma is proved in Section 7.

By setting in (39), $\theta = h_{\infty,m,k}$, defined in (40), we obtain for all $(m, k) \in \mathbb{Z}^2$,

$$Y^{(q,K)}_{m,k} = Z^{(K)}_{q,d}(h_{\infty,m,k}).$$

Hence the right-hand side of (36) are the wavelet coefficients of the generalized process $Z^{(K)}_{q,d}$ with respect to the wavelet family $\{h_{\infty,m,k}, m, k \in \mathbb{Z}\}$.

In the special case $q = 1$ (Gaussian case), this result corresponds to that of Theorem 1(b) and Remark 5 in [1], obtained in the case where $\gamma_j = 2^j$. In this special case, we have $Z^{(K)}_{1,d} = B_{(d+K)}$, where $B_{(d)}$ is the centered generalized Gaussian process such that for all $\theta_1, \theta_2 \in S^{(0)}_{1,d}$,

$$\text{Cov}(B_{(d)}(\theta_1), B_{(d)}(\theta_2)) = \int_{\mathbb{R}} |\lambda|^{-2d} \bar{\theta}_1(\lambda) \bar{\theta}_2(\lambda) \text{d}\lambda.$$

It is interesting to observe that, under additional assumptions on $\theta$, for $K \geq 1$, $Z^{(K)}_{q,d}(\theta)$ can also be defined by

$$Z^{(K)}_{q,d}(\theta) = \int_{\mathbb{R}} \bar{Z}^{(K)}_{q,d}(\theta)(t) \text{d}t,$$

where $\{\bar{Z}^{(K)}_{q,d}(t), t \in \mathbb{R}\}$ denotes a measurable continuous time process defined by

$$\bar{Z}^{(K)}_{q,d}(t) = \int_{\mathbb{R}^q} e^{i(u_1 + \cdots + u_q)t} - \sum_{\ell=0}^{K-1} \frac{(i(u_1 + \cdots + u_q))^\ell}{\ell!} \text{d}\hat{W}(u_1) \cdots \text{d}\hat{W}(u_q), t \in \mathbb{R}. $$
If, in (41) we set $K = 1$, we recover the usual Hermite process as defined in [19] which has stationary increments. The process $\tilde{Z}_{q,d}^{(0)}(t)$ can be regarded as the Hermite process $\tilde{Z}_{q,d}^{(1)}(t)$ integrated $K - 1$ times. In the special case where $K = q = 1$, we recover the Fractional Brownian Motion $\{B_H(t)\}_{t \in \mathbb{R}}$ with Hurst index $H = d + 1/2 \in (1/2, 1)$.

In the case $K = 0$ we cannot define a random process $Z_{q,d}^{(0)}(t)$ as in (42). The case $K = 0$ would correspond to the derivative of the Hermite process $\tilde{Z}_{q,d}^{(1)}(t)$ but the Hermite process is not differentiable and thus the process $\tilde{Z}_{q,d}^{(0)}(t)$, $t \in \mathbb{R}$ is not defined. When $K = 0$ one can only consider the generalized process $\tilde{Z}_{q,d}^{(0)}(\theta)$. Relation (42) can be viewed as resulting from (39) and (41) by interverting formally the integral signs.

We now state sufficient conditions on $\theta$ for (41) to hold.

**Lemma 5.2.** Let $q$ be a positive integer such that $0 < q < 1/(1 - 2d)$ and $K \geq 1$. Suppose that $\theta \in S_{q,d}^{(K)}$ is complex valued with at least $K$ vanishing moments, that is,

$$\int_{\mathbb{R}} \theta(t) t^\ell \, dt = 0 \quad \text{for all} \quad \ell = 0, 1, \ldots, K - 1 .$$

(43)

Suppose moreover that

$$\int_{\mathbb{R}} |\theta(t)| |t|^{K+(d-1/2)q} \, dt < \infty .$$

(44)

Then Relation (41) holds.

This lemma is proved in Section 7.

If, for example, the $h_j$ are derived from a compactly supported multiresolution analysis then $h_\infty$ will have compact support and so $h_{\infty,m,k}$ will satisfy (44). In this case, the limits $Y_{m,k}^{(q,K)}$ in Theorem 5.1 can therefore be interpreted, for $m, k \in \mathbb{Z}$ as the wavelet coefficients of the process $Z_{q,d}^{(K)}$ belonging to the $q$-th chaos. This interpretation is a useful one even when the technical assumption (44) is not satisfied.

**Self-similarity.**

The processes $Z_{q,d}^{(K)}$ and $\tilde{Z}_{q,d}^{(K)}$ are self-similar. Self-similarity can be defined for processes indexed by $t \in \mathbb{R}$ as well as for generalized processes indexed by functions $\theta$ belonging to some suitable space $\mathcal{S}$, for example the space $S_{q,d}^{(K)}$ defined above.

A process $\{Z(t), t \in \mathbb{R}\}$ is said to be self-similar with parameter $H > 0$ if for any $a > 0$,

$$\{a^H Z(t/a), t \in \mathbb{R}\} \overset{\text{fd}}{=} \{Z(t), t \in \mathbb{R}\} ,$$

where the equality holds in the sense of finite-dimensional distributions. A generalized process $\{Z(\theta), \theta \in \mathcal{S}\}$ is said to be self-similar with parameter $H > 0$ if for any $a > 0$ and $\theta \in \mathcal{S}$,

$$Z(a^H \theta) \overset{\text{d}}{=} Z(\theta) ,$$

where the equality holds in the sense of finite-dimensional distributions.
where \( \theta^{a,H}(u) = a^{-H}\theta(u/a) \) (see [17], Page 5). Here \( S \) is assumed to contain both \( \theta^{a,H} \) and \( \theta \).

Observe that the process \( \{\tilde{Z}^{(K)}_{q,d}(t), t \in \mathbb{R}\} \), with \( K \geq 1 \) is self-similar with parameter

\[
H = K + qd - q/2 = (K - 1) + (d(q) + 1/2) \, .
\]

As noted above \( \tilde{Z}^{(K)}_{q,d} \) can be regarded as \( \tilde{Z}^{(1)}_{q,d} \) integrated \( K - 1 \) times.

The generalized process \( \{Z^{(K)}_{q,d}(\theta), \theta \in S^{(K)}_{q,d}\} \), which is defined in (39) with \( K \geq 0 \), is self-similar with the same value of \( H \) as in (45), but this time the formula is also valid for \( K = 0 \).

In particular, the Hermite process \( (K = 1) \) is self-similar with \( H = d(q) + 1/2 \in (1/2, 1) \) and the generalized process \( Z^{(0)}_{q,d}(\theta) \) with \( K = 0 \) is self-similar with \( H = d(q) - 1/2 \in (-1/2, 0) \).

Interpretation of the result.

In view of the preceding discussion, the wavelet coefficients of the subordinated process \( Y \) behave at large scales \( (\gamma_j \to \infty) \) as those of a self-similar process \( Z^{(K)}_{q,d} \) living in the chaos of order \( q_0 \) (the Hermite rank of \( G \)) and with self-similar parameter \( K + d(q_0) - 1/2 \).

6. Proof of Theorem 5.1

**Notation.** It will be convenient to use the following notation. We denote by \( \Sigma_q, q \geq 1, \) the \( \mathbb{C}^q \to \mathbb{C} \) function defined, for all \( y = (y_1, \ldots, y_q) \) by

\[
\Sigma_q(y) = \sum_{i=1}^q y_i \, .
\]

With this notation \( Y^{(q,K)}_{m,k} \) in Theorem 5.1 can be expressed as

\[
Y^{(q,K)}_{m,k} = (\gamma_m)^{1/2} \iint_{\mathbb{R}^q} \exp \circ \Sigma_q(i\xi) \frac{\exp(\Sigma_q(i\xi)\xi)}{(|\xi_1| \cdots |\xi_q|d)} \cdot d\hat{W}(\xi_1) \cdots d\hat{W}(\xi_q) \, .
\]

where \( \circ \) denotes the composition of functions.

We will separate the Wiener chaos expansion (28) of \( W_{j,k} \) into two terms depending on the position of \( q \) with respect to \( q_c \). The first term includes only the \( q \)'s for which \( H_q(x) \) exhibits long-range dependence (LD), that is,

\[
W^{(LD)}_{j,k} = \sum_{q=0}^{q_c} \frac{c_q}{q!} W^{(q)}_{j,k} \, ,
\]
and the second term includes the terms which exhibit short–range dependence (SD)
\[
W_{j,k}^{(SD)} = \sum_{q=q_c+1}^{\infty} \frac{C_q}{q!} W_{j,k}^{(q)} .
\] (48)

Using Representation (14) and (27) since \( X \) has unit variance, one has for any \( \ell \in \mathbb{Z} \),
\[
H_q(X_\ell) = H_q \left( \int_{-\pi}^{\pi} e^{i\ell \xi} f^{1/2}(\xi) d\widehat{W}(\xi) \right)
= \int_{(-\pi,\pi)^q} \exp \circ \Sigma_q(i\ell \xi) \times (f^{\circ q}(\xi))^{1/2} \ d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q) .
\]

Then by (29), (10) and (9), we have
\[
W_{j,k}^{(q)} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) H_q(X_\ell)
= \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) \int_{(-\pi,\pi)^q} \exp \circ \Sigma_q(i\ell \xi) \times (f^{\circ q}(\xi))^{1/2} \ d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q)
= \int_{(-\pi,\pi)^q} e^{\Sigma_q(i\gamma_j \xi)} \left( \sum_{m \in \mathbb{Z}} h_j^{(K)}(m) \exp \circ \Sigma_q(-im \xi) \right) (f^{\circ q}(\xi))^{1/2} \ d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q)
= \int_{(-\pi,\pi)^q} e^{\Sigma_q(i\gamma_j \xi)} \left( \hat{h}_j^{(K)} \circ \Sigma_q(\xi) \right) (f^{\circ q}(\xi))^{1/2} \ d\widehat{W}(\xi_1) \cdots d\widehat{W}(\xi_q) .
\]

Then
\[
W_{j,k}^{(q)} = \hat{I}_q(f_{j,k}^{(q)}) ,
\] (49)
with
\[
f_{j,k}^{(q)}(\xi) = (\exp \circ \Sigma_q(ik \gamma_j \xi)) \left( \hat{h}_j^{(K)} \circ \Sigma_q(\xi) \right) (f^{\circ q}(\xi))^{1/2} \times \mathbb{1}_{(-\pi,\pi)}^{\circ q}(\xi) ,
\]
where \( \xi = (\xi_1, \ldots, \xi_q) \) and \( f^{\circ q}(\xi) = f(\xi_1) \cdots f(\xi_q) \).

The two following results provide the asymptotic behavior of each term of the sum in (47) and of \( W_{j,k}^{(SD)} \), respectively. They are proved in Sections 6.1 and 6.2, respectively. The first result concerns the terms with long memory, that is, with \( q \leq q_c \). The second result concerns the terms with short memory for which \( q > q_c \).

**Proposition 6.1.** Suppose that \( q \in \{1, \ldots, q_c\} \). Then, as \( j \to \infty \),
\[
\left( \gamma_j^{-(d(q)+K)} W_{j+m,k}^{(q)} \right)_{m, k \in \mathbb{Z}} \xrightarrow{\text{law}} \left( (f^*(0))^{q/2} Y_{m,k}^{(q,K)} \right)_{m, k \in \mathbb{Z}} ,
\] (50)
where \( Y_{m,k}^{(q,K)} \) is given by (37).
Proposition 6.2. We have, for any \( k \in \mathbb{Z} \), as \( j \to \infty \),
\[
W_{j+m,k}^{(SD)} = O_P(\gamma_j^K) .
\] (51)

It follows from Proposition 6.1 that the dominating term in (47) is given by the chaos of order \( q = q_0 \). Now, since \( d(q_0) > 0 \) by (32), we get from Proposition 6.2 that, for all \((k, m)\), as \( j \to \infty \),
\[
W_{j+m,k}^{(SD)} = o_p(\gamma_j^d(q_0)+K) .
\]

This concludes the proof of Theorem 5.1.

6.1. Proof of Proposition 6.1

We first express the distribution of \( \{W_{j+m,k}^{(q)}, m, k \in \mathbb{Z}\} \) as a finite sum of stochastic integrals and then show that each integral converges in \( L^2(\Omega) \).

Lemma 6.1. Let \( q \in \mathbb{N}^* \). For any \( j \)
\[
W_{j+m,k}^{(q)} \stackrel{(fidi)}{=} \sum_{s=-\lfloor q/2 \rfloor}^{\lceil q/2 \rceil} W_{m,k}^{(j,q,s)} ,
\] (52)
where \([a] \) denotes the integer part of \( a \), and for any \( q \in \mathbb{N}^* \), \( s \in \mathbb{Z} \),
\[
W_{m,k}^{(j,q,s)} = \int_{\zeta \in \mathbb{R}^q} 1_{\Gamma(q,s)}(\zeta_j^{-1}) f_{m,k}(\zeta; j, q) \ d\hat{W}(\zeta_1) \cdots d\hat{W}(\zeta_q) ,
\] (53)
where \( f_{m,k}(\zeta; j, q) \) is defined by (setting \( \xi = \zeta_j^{-1} \))
\[
f_{m,k}(\gamma_j \xi; j, q) = \gamma_j^{-q/2} \exp \circ \Sigma_q(i \zeta_j m k \xi) \times \hat{h}_{j+m} \circ \Sigma_q(\xi) \left\{ 1 - \exp \circ \Sigma_q(-i \xi) \right\}^K (f^{\otimes q}(\xi))^{1/2} .
\] (54)

and where
\[
\Gamma(q,s) = \left\{ \xi \in (-\pi, \pi]^q, -\pi + 2s\pi < \sum_{i=1}^q \xi_i \leq \pi + 2s\pi \right\} .
\] (55)

Proof. Using (49), with \( j \) replaced by \( j + m \), and (9), we get
\[
W_{j+m,k}^{(q)} = \int_{(-\pi, \pi]^q} \exp \circ \Sigma_q(i \gamma_j m k \xi) \frac{\hat{h}_{j+m} \circ \Sigma_q(\xi)}{\left\{ 1 - \exp \circ \Sigma_q(-i \xi) \right\}^K} (f^{\otimes q}(\xi))^{1/2} d\hat{W}(\xi_1) \cdots d\hat{W}(\xi_q) .
\]

By (54), we thus get
\[
W_{j+m,k}^{(q)} = \int_{\xi \in (-\pi, \pi]^q} \gamma_j^{q/2} f_{m,k}(\gamma_j \xi; j, q) d\hat{W}(\xi_1) \cdots d\hat{W}(\xi_q)
\] (56)

\[
\stackrel{(fidi)}{=} \int_{\xi \in (-\gamma_j \pi, \gamma_j \pi]^q} f_{m,k}(\zeta; j, q) d\hat{W}(\xi_1) \cdots d\hat{W}(\xi_q) ,
\]

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where we set $\zeta = \gamma_j \xi$ (see Theorem 4.4 in [17]). Observe that for all $\zeta \in (-\gamma_j \pi, \gamma_j \pi]^q$, 

$$-\pi \gamma_j - 2[q/2] \pi \gamma_j \leq -q \gamma_j \pi \leq \sum_{i=1}^{q} \zeta_i \leq q \gamma_j \pi \leq \pi \gamma_j + 2[q/2] \pi \gamma_j$$.

The result follows by using that for any $\zeta \in (-\gamma_j \pi, \gamma_j \pi]^q$, there is a unique $s = -[q/2], \ldots, [q/2]$ such that $\zeta/\gamma_j \in \Gamma(q,s)$. 

**Proof of Proposition 6.1.** In view of Lemma 6.1, we shall look at the $L^2(\Omega)$ convergence of the normalized $W_{m,k}^{(j,q,s)}$ at each value of $s$. Proposition 6.1 will follow from the following convergence results, valid for all fixed $m, k \in \mathbb{Z}$ as $j \to \infty$. For $s = 0$, 

$$\gamma_j^{-(d(q)+K)} W_{m,k}^{(j,q,0)} L^2 \to (f^*(0))^{q/2} Y_{m,k}^{(q,K)}, \quad (57)$$

whereas for other values of $s$, namely for all $s \in \{-[q/2], \ldots, -1, 1, \ldots, [q/2]\}$, 

$$\gamma_j^{-(d(q)+K)} W_{m,k}^{(j,q,s)} L^2 \to 0, \quad (58)$$

where $d(q)$ is defined in [31].

We now prove these convergence using the representation (53). By (1) and $|1 - e^{i\lambda}| \geq 2|\lambda|/\pi$ on $\lambda \in (-\pi, \pi)$, we have that 

$$f(\lambda) \leq \left(\frac{\pi}{2}\right)^{-2d} \|f^*\|_{\infty} |\lambda|^{-2d}, \quad \lambda \in [-\pi, \pi]. \quad (59)$$

By definition of $\Gamma(q,s)$ in (55), we have, for all $\zeta \in \gamma_j \Gamma(q,s)$, $\gamma_j^{-1} \sum_i \zeta_i - 2\pi s \in (-\pi, \pi]$. Hence using the $(2\pi)$-periodicity of $\hat{h}_{j+m}$, we can use (1) for bounding $\hat{h}_{j+m}(\gamma_j^{-1} \sum_i \zeta_i)$. With the change of variables $\zeta = \gamma_j \xi$ and (59), for all $\zeta \in \gamma_j \Gamma(q,s)$ and $j$ large enough so that $\gamma_{j+m}/\gamma_j \geq \pi m/2$, 

$$\gamma_j^{-(d(q)+K)} |f_{m,k}(\zeta; j, q)| = \gamma_j^{-(dq-q/2+1/2+K)} |f_{m,k}(\zeta; j, q)| \leq C_0 \|g(\zeta; 2\pi \gamma_j s)\|, \quad (60)$$

where $C_0$ is a positive constant and 

$$g(\zeta; t) = \left(1 + \left|\sum_{i=1}^{q} \zeta_i - t\right|\right)^{-\alpha-K} \prod_{i=1}^{q} |\zeta_i|^{-d}.$$

The squared $L^2$-norm of $g(\cdot; t)$ reads 

$$J(t) = \int_{\mathbb{R}^d} g^2(\zeta; t) \, d\zeta = \int_{\mathbb{R}^q} \left(1 + \left|\sum_{i=1}^{q} \zeta_i - t\right|\right)^{-2\alpha-2K} \prod_{i=1}^{q} |\zeta_i|^{-2d} \prod_{i=1}^{q} d\zeta_i.$$

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We now show that Lemma 8.4 applies with $M_1 = 2\alpha + 2K$, $M_2 = 0$ and $\beta_i = 2d$ for $i = 1, \ldots, q$. Indeed, we have $M_2 - M_1 = -2\alpha - 2K \leq -2\alpha < -1$. Further, for all $\ell = 1, \ldots, q - 1$, we have, by the assumption on $d$,

$$
\sum_{i=\ell}^{q} \beta_i = 2d(1 + q - \ell) > (1 + q - \ell)(1 - 1/q) = q - \ell + (\ell - 1)/q \geq q - \ell.
$$

Finally, since $\alpha > 1/2$, one has $M_2 - M_1 + q = -2\alpha - 2K + q < q - 1 \leq \sum_i \beta_i$.

Applying Lemma 8.4, we get $J(t) \to 0$ as $|t| \to \infty$ and $J(0) < \infty$. Thus, if $s \neq 0$, one has $t = 2\pi \gamma s \to \infty$ as $j \to \infty$ and hence we obtain (55). If $s = 0$, then $t = 2\pi \gamma s = 0$ and using the bound (60), $J(0) < \infty$, and the dominated convergence theorem, we have that the convergence (57) follows from the convergence at a.e. $\zeta \in \mathbb{R}^q$ of the left hand side of (60), which we now establish. Recall that $f_{m,k}$ is defined in (54). By (6), (1) and the continuity of $f^*$ at the origin, we have, as $j \to \infty$,

$$
\gamma_j^{-1/2} \tilde{h}_{j+m} \circ \Sigma_q (\zeta/\gamma_j) = \left( \gamma_j^{1/2} \tilde{h}_{j+m} \circ \Sigma_q (\zeta/\gamma_j) \right) \frac{1/2}{\gamma_j^{1/2}} \tilde{h}_\infty(\Sigma_m (\zeta_1 + \cdots + \zeta_q)),
$$

and for every $\ell = 1, \ldots, q$

$$
\gamma_j^{-2d} f(\zeta_1/\gamma_j) = \gamma_j^{-2d} \left| 1 - e^{-i\zeta_i/\gamma_j} \right|^{-2d} f^*(\zeta_i/\gamma_j) \to f^*(0) |\zeta_i|^{-2d}.
$$

Hence $\gamma_j^{-(d(q)+K)} f_{m,k}(\zeta; j, q, 0) \mathbb{I}_{\Gamma(q,s)}(\gamma_j^{-1} \zeta)$ converges to

$$
(\Sigma_m)^{1/2}(f^*(0))^{q/2} \frac{e^{it(\Sigma_m(\zeta_1 + \cdots + \zeta_q))} \times \hat{h}_\infty(\Sigma_m (\zeta_1 + \cdots + \zeta_q))}{(it(\zeta_1 + \cdots + \zeta_q))^{K} |\zeta_1|^d \cdots |\zeta_q|^d}.
$$

This concludes the proof. \hfill \Box

### 6.2 Proof of Proposition 6.2

We now consider the short-range dependence part of the wavelet coefficients ($W_{j,k}$) defined by (29) and (18). These wavelet coefficients can be equivalently defined as

$$
W_{j,k}^{(SD)} = \sum_{\ell \in \mathbb{Z}} h_{j+K}^\ell(\gamma_j k - \ell) \Delta^K Y_\ell^{(SD)},
$$

where we have set

$$
\Delta^K Y_\ell^{(SD)} = \sum_{q \geq q_0 + 1} \frac{\zeta_q}{q!} H_q(X_\ell), \quad \ell \in \mathbb{Z}.
$$

Using Lemma 4.1 since (26) holds and the processes $\{H_q(X_\ell)\}_{\ell \in \mathbb{Z}}$ are uncorrelated weakly stationary, the process $\{\Delta^K Y_\ell^{(SD)}\}_{\ell \in \mathbb{Z}}$ is weakly stationary with spectral density

$$
f^{(SD)}(\lambda) = \sum_{q \geq q_0 + 1} \frac{\zeta_q^2}{q^4} f^{(*q)}(\lambda), \quad \lambda \in (-\pi, \pi).
$$
By Lemma \textbf{(4.1)}(ii), we have that \( \|f^{(\ast\{q-1\})}\|_\infty < \infty \). Using that \( \|g_1 \ast g_2\|_\infty \leq \|g_1\|_\infty \|g_2\|_1 \) and \( \|f\|_1 = 1 \) by assumption, an induction yields

\[
\sup_{q>q_c} \|f^{(\ast q)}\|_\infty \leq \|f^{(\ast\{q-1\})}\|_\infty .
\]

Hence, by \textbf{(26)}, we get \( \|f^{(SD)}\|_\infty < \infty \). It follows that, for \( W^{(SD)}_{j,k} \) defined in \textbf{(61)}, there is a positive constant \( C \) such that,

\[
\mathbb{E}[W^{(SD)^2}_{j,k}] \leq \|f^{(SD)}\|_\infty \int_{-\pi}^\pi \left| \hat{h}_j^{(K)}(\lambda) \right|^2 d\lambda \leq C \int_{-\pi}^\pi \left| \lambda \right|^{-2K} \left| \hat{h}_j^{(K)}(\lambda) \right|^2 d\lambda = O(\gamma_j^{2K}) ,
\]

where we used \textbf{(1)} with \( M \geq K \) and \( \alpha > 1/2 \). This last relation implies \textbf{(51)} and concludes the proof of Proposition \textbf{6.2} \hfill \Box

7. Proof of Lemmas \textbf{5.1} and \textbf{5.2}

7.1. Proof of Lemma \textbf{5.1}

Let us first prove that if \( \theta \in S_{q,d}^{(K)} \) then \( Z_{q,d}^{(K)}(\theta) \) exists. Indeed, by Definition \textbf{5.1}, \( Z_{q,d}^{(K)}(\theta) \) exists if

\[
\int_{\mathbb{R}^q} \frac{\hat{\theta}(u_1 + \cdots + u_q)^2}{|u_1 + \cdots + u_q|^{2K}|u_1 \cdots u_q|^{2d}} du_1 \cdots du_q < \infty . \tag{62}
\]

Use now Lemma \textbf{3.3} with \( \beta_1 = \cdots = \beta_q = -2d \) and \( f(x) = |\hat{\theta}(x)|^2/|x|^{2K} \) and deduce that Condition \textbf{(62)} is equivalent to

\[
\Gamma \int_{\mathbb{R}} |\hat{\theta}(s)|^2 |s|^{q-2d-2K} ds < \infty , \tag{63}
\]

where

\[
\Gamma = \prod_{i=2}^q \left( \int_{\mathbb{R}} |t|^{q-2d(q-i+1)}|1-t|^{-2d} dt \right) .
\]

Note that the conditions \( 0 < d < 1/2 \) and \( 0 < q < 1/(1-2d) \) ensure that \( \Gamma \) is finite. Further, Relation \textbf{(63)} implies \( \theta \in S_{q,d}^{(K)} \).

We now prove that for any \( m, k, h_{\infty,m,k} \in S_{q,d}^{(K)} \) when \( K \in \{0, \ldots, M\} \). By Definition \textbf{(10)} of \( h_{\infty,m,k} \)

\[
\hat{h}_{\infty,m,k}(\xi) = (\gamma_m)^{1/2} e^{-|\gamma_m\xi|} \hat{h}_{\infty}(\gamma_m \xi) .
\]

Hence

\[
\int_{\mathbb{R}} |\hat{h}_{\infty,m,k}(s)|^2 |s|^{q-1-2qd-2K} ds = \gamma_m \int_{\mathbb{R}} |\hat{h}_{\infty}(\gamma_m s)|^2 |s|^{q-1-2qd-2K} ds .
\]

Set \( v = \gamma_m s \) and deduce that \( h_{\infty,m,k} \in S_{q,d}^{(K)} \) is equivalent to

\[
\gamma_m^{2-(q-1-2qd-2K)} \int_{\mathbb{R}} |\hat{h}_{\infty}(v)|^2 |v|^{q-1-2qd-2K} dv < \infty .
\]
Assumption (12) implies that
\[
\int_{\mathbb{R}} |\hat{h}_\infty(v)|^2 |v|^{q-1-2qd-2K} dv \leq \int_{\mathbb{R}} \frac{|v|^{2M}}{(1+|v|)^{2M+2\alpha}} |v|^{q-1-2qd-2K} dv.
\]
Since \( M \geq K \) and \( q(1-2d) \in (0, 1) \) then \( 2M+q-1-2qd-2K = (2M-2K)+q(1-2d)-1 > -1 \). Further \( \alpha > 1/2 \) and \( q(1-2d) \in (0, 1) \) imply that \( 2M-2M-2\alpha+(q-1-2qd-2K) = -2\alpha - 2K + q(1-2d) - 1 < -1 \). Then
\[
\int_{\mathbb{R}} |\hat{h}_{\infty,m,k}(s)|^2 |s|^{q-1-2qd-2K} ds < \infty.
\]
holds and \( h_{\infty,m,k} \in S_{q,d}^{(K)} \).

7.2. Proof of Lemma 5.2

Let \( a_t(u_1, \ldots, u_q) \) denote the kernel of the integral in (12) defining \( \tilde{Z}_{q,d}^{(K)} \) and suppose we can exchange the order of integration and write
\[
\int_{\mathbb{R}^q} \tilde{Z}_{q,d}^{(K)}(t) \theta(t) dt = \int_{\mathbb{R}^q} \left[ \int_{\mathbb{R}} a_t(u_1, \ldots, u_q) \theta(t) dt \right] d\tilde{W}(u_1) \cdots d\tilde{W}(u_q). \tag{64}
\]
Then condition (43) gives
\[
\int_{\mathbb{R}} \left[ e^{it(u_1+\cdots+u_q)} - \sum_{\ell=0}^{K-1} \frac{(it(u_1+\cdots+u_q))^n}{\ell!} \right] \theta(t) dt = \int_{\mathbb{R}} e^{it(u_1+\cdots+u_q)} \theta(t) dt = \theta \circ \Sigma_q(u),
\]
showing that (64) equals \( \tilde{Z}_{q,d}^{(K)}(\theta) \) defined in (39). It remains to justify the change of order of integration in (64) by using a stochastic Fubini theorem, (see for instance [20, Theorem 2.1]). A sufficient condition is
\[
\int_{\mathbb{R}} \left( a_t^2(u_1, \ldots, u_q) du_1 \cdots du_q \right)^{1/2} dt < \infty.
\]
This condition is satisfied, because setting \( v = tu \), we have
\[
\int_{\mathbb{R}^q} \left| e^{it(u_1+\cdots+u_q)} - \sum_{\ell=0}^{K-1} \frac{(it(u_1+\cdots+u_q))^\ell}{\ell!} \right|^2 |i(u_1+\cdots+u_q)|^{-2K} |u_1 \cdots u_q|^{-2d} du_1 \cdots du_q \leq |v|^{2K+2d-2q} \int_{\mathbb{R}^q} (1+|u_1+\cdots+u_q|)^{-2K} |u_1 \cdots u_q|^{-2d} du_1 \cdots du_q. \]
8. Auxiliary lemmas

The following lemma provides a bound for the convolution of two functions exploding at the origin and decaying polynomially at infinity.

**Lemma 8.1.** Let $\alpha > 1$ and $\beta_1, \beta_2 \in [0,1)$ such that $\beta_1 + \beta_2 < 1$, and set

$$g_t(t) = |t|^{-\beta_1}(1 + |t|)^{\beta_2 - \alpha}.$$  

Then

$$\sup_{u \in \mathbb{R}} \left( (1 + |u|)^{\alpha} \int_{\mathbb{R}} g_1(u - t)g_2(t) \, dt \right) < \infty.$$  

**Proof.** We first show that

$$J(u) = \int_{\mathbb{R}} g_1(u - t)g_2(t) \, dt = \int_{\mathbb{R}} |u - t|^{-\beta_1}(1 + |u - t|)^{\beta_1 - \alpha} |t|^{-\beta_2}(1 + |t|)^{\beta_2 - \alpha} \, dt$$

is uniformly bounded on $\mathbb{R}$. Using the assumptions on $\beta_1, \beta_2$, there exist $p > 1$ such that $\beta_1 < 1/p < 1 - \beta_2$. Let $q$ be such that $1/p + 1/q = 1$. The Hölder inequality implies that

$$J(u)^{pq} \leq \int_{\mathbb{R}} |t|^{-p\beta_1}(1 + |t|)^{p\beta_1 - p\alpha} \, dt \times \int_{\mathbb{R}} |t|^{-q\beta_2}(1 + |t|)^{q\beta_2 - q\alpha} \, dt.$$  

The condition on $\alpha, \beta_1, \beta_2, p$ and the definition of $q$ imply that these two integrals are finite. Hence $\sup_u J(u) < \infty$.

We now determine how fast $J(u)$ tends to 0 as $u \to \infty$. Observe that, if $|t - u| \leq |u|/2$, then $|t| \geq |u|/2$. By splitting the integral in two integrals on the domains $|t - u| \leq |u|/2$ and $|t - u| > |u|/2$, we get $J(u) \leq J_1(u) + J_2(u)$ with

$$J_1(u) \leq (|u|/2)^{-\beta_2}(1 + |u|/2)^{\beta_2 - \alpha} \int_{\mathbb{R}} |u - t|^{-\beta_1}(1 + |t - u|)^{\beta_1 - \alpha} \, dt,$$

and

$$J_2(u) \leq (|u|/2)^{-\beta_1}(1 + |u|/2)^{\beta_1 - \alpha} \int_{\mathbb{R}} |t|^{-\beta_2}(1 + |t|)^{\beta_2 - \alpha} \, dt.$$  

Now, as $|u| \to \infty$, we have $J_i(u) = O(|u|^{-\alpha})$ for $i = 1, 2$, which achieves the proof. \hfill $\Box$

The next lemma describes the convolutions of two periodic functions that explode at the origin as a power. A different definition of convolution is involved here (see (33)).

**Lemma 8.2.** Let $(\beta_1, \beta_2) \in (0,1)^2$. Let $g_1, g_2$ be $(2\pi)$-periodic functions such that $g_i(\lambda) = |\lambda|^{-\beta_i} g_i^*(\lambda), i = 1, 2$. Each $g_i^*(\lambda)$ is a $(2\pi)$-periodic non-negative function, bounded on $(-\pi, \pi)$ and positive at the origin, where it is also continuous. Let $g = g_1 \ast g_2$ as defined in (33). Then,

- If $\beta_1 + \beta_2 < 1$, $g$ is bounded and continuous on $(-\pi, \pi)$, and satisfies $g(0) > 0$.  

If \( \beta_1 + \beta_2 > 1 \),
\[
g(\lambda) = |\lambda|^{-(\beta_1 + \beta_2 - 1)} g^*(\lambda) ,
\]
where \( g^*(\lambda) \) is bounded on \((-\pi, \pi)\) and converges to a positive constant as \( \lambda \to 0 \). If moreover for some \( \beta \in (0, 2] \) such that \( \beta < \beta_1 + \beta_2 - 1 \) and some \( L > 0 \), one has for any \( i \in \{1, 2\} \)
\[
|g_i^*(\lambda) - g_i^*(0)| \leq L|\lambda|^\beta, \quad \forall \lambda \in (-\pi, \pi) ,
\]
then there exists some \( L' > 0 \) depending only on \( L, \beta_1, \beta_2 \) such that
\[
|g^*(\lambda) - g^*(0)| \leq L'|\lambda|^\beta, \quad \forall \lambda \in (-\pi, \pi) .
\]

**Proof.** By (33) and (2π)-periodicity, we may write
\[
g(\lambda) = \int_{-\pi}^{\pi} g_1(u) g_2(\lambda - u) \, du = \int_{-\pi}^{\pi} |\lambda - u|^{-\beta_1} g_1^*(\lambda - u) |u|^{-\beta_2} g_2^*(u) \, du . \tag{67}
\]

Let us first consider the case \( \beta_1 + \beta_2 < 1 \). We clearly have \( g(0) > 0 \). To prove that \( g \) is bounded, we proceed as in the case of convolutions of non-periodic functions (see the proof of Lemma 8.1), namely, for \( p, q \) such that \( \beta_1 < 1/p < 1 - \beta_2 \) and \( 1/p + 1/q = 1 \), the Hölder inequality gives that
\[
\|g\|_p^q \leq \|g_1\|_p^p \|g_2\|_q^q \leq \|g_1^*\|_\infty^p \|g_2^*\|_\infty^q \int_{-\pi}^{\pi} |t|^{-p\beta_1} \, dt \times \int_{-\pi}^{\pi} |t|^{-q\beta_2} \, dt < \infty . \tag{68}
\]

For any \( \epsilon > 0 \) and \( i = 1, 2 \), let \( g_{e,i} \) be the (2π)-periodic function such that for all \( \lambda \in (-\pi, \pi) \),
\[
g_{e,i}(\lambda) = \mathbb{1}_{(-\epsilon, \epsilon)}(\lambda) \, g_i(\lambda) \quad \text{and let} \quad \bar{g}_{e,i} = g_i - g_{e,i} \,. \quad \text{Then} \quad g = \bar{g}_{e,1} \ast \bar{g}_{e,2} + g_{e,1} \ast \bar{g}_{e,2} + \bar{g}_{e,1} \ast g_{e,2} + g_{e,1} \ast g_{e,2} \,. \quad \text{Since} \quad \bar{g}_{e,i} \quad \text{is bounded for} \quad i = 1, 2, \quad \text{we have} \quad \bar{g}_{e,1} \ast \bar{g}_{e,2} \quad \text{is continuous}. \quad \text{On the other hand, using the Hölder inequality as in (68), we get that} \quad \|g_{e,1} \ast \bar{g}_{e,2}\|_\infty, \quad \|\bar{g}_{e,1} \ast g_{e,2}\|_\infty, \quad \|\bar{g}_{e,1} \ast \bar{g}_{e,2}\|_\infty \quad \text{tend to zero as} \quad \epsilon \to 0. \quad \text{Hence} \quad g \quad \text{is continuous as well}. \]

We now consider the case \( \beta_1 + \beta_2 \geq 1 \). Setting \( v = u/\lambda \) in (67), we get, for any \( \lambda \in [-\pi, \pi] \setminus \{0\} \),
\[
g^*(\lambda) = |\lambda|^{\beta_1 + \beta_2 - 1} g(\lambda) = \int_\mathbb{R} \mathbb{1}_{(-\pi/|\lambda|, \pi/|\lambda|)}(v) \, \{(1 - v)\}_{\lambda}^{\beta_1} |v|^{-\beta_2} g_1^*(\lambda(1-v)) g_2^*(\lambda v) \, dv , \]
where for any real number \( x \) and \( \lambda \neq 0 \), \( \{x\}_{\lambda} \) denotes the unique element of \([-\pi/|\lambda|, \pi/|\lambda|]\) such that \( x - \{x\}_{\lambda} \in \mathbb{Z} \). Take now \( |\lambda| \) small enough so that \( \pi/|\lambda| > 2 \). Then, for any \( v \in (-\pi/|\lambda| + 1, \pi/|\lambda|) \), we have \( \{(1-v)\}_{\lambda} = |1-v| \geq |1-|v|| \) and, for any \( v \in (-\pi/|\lambda|, -\pi/|\lambda| + 1) \), we have
\[
\{(1-v)\}_{\lambda} = |1-v - 2\pi/|\lambda|| = 2\pi/|\lambda| + v - 1 \geq -v - 1 = |1-|v|| . \tag{69}
\]
Thus we have \( \mathbb{1}_{(-\pi/|\lambda|, \pi/|\lambda|)}(v) \, \{(1-v)\}_{\lambda}^{-\beta_1} \leq |1-|v||^{-\beta_1} \) for all \( v \in \mathbb{R} \). We conclude that for \( |\lambda| \) small enough, the integrand in the last display is bounded from above by
We set generality and denote random dominated convergence, as \( \lambda \to 0 \),

\[
g^*(\lambda) \to g_1^*(0)g_2^*(0) \int_{\mathbb{R}} |1 - v|^{-\beta_1} |v|^{-\beta_2} \, dv > 0 .
\] (70)

We set \( g^*(0) \) equal to this limit.

Suppose moreover that \( g_1^*, g_2^* \) satisfy (66). We take \( g_1^*(0) = g_2^*(0) = 1 \) without loss of generality and denote \( r_i(\lambda) = |g_i^*(\lambda) - 1| \) for \( i = 1, 2 \). Then \( r(\lambda) = |g^*(\lambda) - g^*(0)| \), where \( g^*(0) \) is defined as the limit in (70), is at most

\[
\int_{\mathbb{R}} \left| \mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v) \right| \{(1 - v)\} \lambda^{-\beta_1} |v|^{-\beta_2} g_1^*(\lambda(1 - v))g_2^*(\lambda v) - |1 - v|^{-\beta_1} |v|^{-\beta_2} \, dv .
\]

Setting \( g_i^*(\lambda) = (g_i^*(\lambda) - 1) + 1 \), we have \( r \leq A + B_1 + B_2 + C \) with

\[
A(\lambda) = \int_{\mathbb{R}} \left| \mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v) \right| \{(1 - v)\} \lambda^{-\beta_1} |v|^{-\beta_2} - |1 - v|^{-\beta_1} |v|^{-\beta_2} \, dv ,
\]

\[
B_i(\lambda) = \int_{\mathbb{R}} \mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v) \{(1 - v)\} \lambda^{-\beta_1} |v|^{-\beta_2} r_i(\lambda v) \, dv ,
\]

where \( (i, j) \) is \((1, 2)\) or \((2, 1)\), and

\[
C(\lambda) = \int_{\mathbb{R}} \mathbb{1}_{(-\pi/|\lambda|,\pi/|\lambda|)}(v) \{(1 - v)\} \lambda^{-\beta_1} |v|^{-\beta_2} r_1(\lambda(1 - v))r_2(\lambda v) \, dv .
\]

Since \( \{(1 - v)\} \lambda = 1 - v \) for \( v \in [-\pi/|\lambda| + 1, \pi/|\lambda|] \) and \( \lambda \) large enough, we have

\[
A(\lambda) = \int_{(-\pi/|\lambda|,\pi/|\lambda|)} [1 - v]^{-\beta_1} |v|^{-\beta_2} \, dv \\
+ \int_{-\pi/|\lambda|}^{-\pi/|\lambda| + 1} \left| \{(1 - v)\} \lambda^{-\beta_1} |v|^{-\beta_2} - |1 - v|^{-\beta_1} |v|^{-\beta_2} \right| \, dv .
\]

The first integral is \( O(|\lambda|^{\beta_1 + \beta_2 - 1}) \). Using (39), the second line of the last display is less than

\[
\int_{\pi/|\lambda|}^{\pi/|\lambda| - 1} \left[ 1 - v |v|^{-\beta_2} + |1 + v|^{-\beta_1} v^{-\beta_2} \right] \, dv = O(|\lambda|^{\beta_1 + \beta_2}) .
\]

We conclude that as \( \lambda \to 0 \), \( A(\lambda) = O(|\lambda|^{\beta_1 + \beta_2 - 1}) \). Moreover using that \( r_i(\lambda) \leq L|\lambda|^\beta \) and \( \beta_1 + \beta_2 - \beta > 1 \), we have \( B_i(\lambda) = O(|\lambda|^\beta) \) for \( i = 1, 2 \). The same is true for \( C \) since \( r_1 \) and \( r_2 \) are also bounded on \( \mathbb{R} \). This achieves the proof.

Lemma 8.3. Let \( p \) be a positive integer and \( f : \mathbb{R} \to \mathbb{R}_+ \). Then, for any \( \beta \in \mathbb{R}^q \),

\[
\int_{\mathbb{R}^q} f(y_1 + \cdots + y_q) \prod_{i=1}^q |y_i|^{\beta_i} \, dy_1 \cdots dy_q = \Gamma \times \int_{\mathbb{R}} f(s)|s|^{-q + \beta_1 + \cdots + \beta_q} \, ds ,
\] (71)
where, for all \( i \in \{1, \ldots, q\} \), \( B_i = \beta_i + \cdots + \beta_q \) and

\[
\Gamma = \prod_{i=2}^{q} \left( \int_{\mathbb{R}} |t|^{q-i+B_i} |1 - t|^{\beta_{i-1}} \, dt \right).
\]

(We note that \( \Gamma \) may be infinite in which case (71) holds with the convention \( \infty \times 0 = 0 \)).

**Proof.** Relation (71) is obtained by using the following two successive change of variables followed by an application of the Fubini Theorem. Setting, for all \( i \)
\( (\Gamma = \prod_{i=2}^{q} \left( \int_{\mathbb{R}} |t|^{q-i+B_i} |1 - t|^{\beta_{i-1}} \, dt \right).)
\]

Let \( \ell \in \mathbb{R} \) and \( q \) be a positive integer. Let \( \beta = (\beta_1, \cdots, \beta_q) \in (-\infty, 1)^q \), \( M_1 > 0 \) and \( M_2 > -1 \) such that \( M_2 - M_1 < -1 \). Assume that \( q + M_2 - M_1 < \sum_{i=1}^{q} \beta_i \), and that for any \( \ell \in \{1, \cdots, q-1\} \), \( \sum_{i=\ell}^{q} \beta_i > q - \ell \). Set for any \( a \in \mathbb{R} \),

\[
J_q(a; M_1, M_2; \beta) = \int_{\mathbb{R}^q} \frac{|\Sigma_q(\zeta) - a|^{M_2}}{(1 + |\Sigma_q(\zeta) - a|)^{M_1} \prod_{i=1}^{q} |\zeta_i|^\beta} \, d\zeta.
\]

Then one has

\[
\sup_{a \in \mathbb{R}} (1 + |a|)^{1-q+\sum_{i=1}^{q} \beta_i} J_q(a; M_1, M_2; \beta) < \infty.
\]
In particular,

\[ J_q(0; M_1, M_2; \beta) < \infty, \]

and

\[ J_q(a; M_1, M_2; \beta) = O(|a|^{-(1/q + \sum_{i=1}^q \beta_i)}) \quad \text{as } a \to \infty. \]

**Proof.** Since \( J_q(a; M_1, M_2; \beta_1, \ldots, \beta_q) = J_q(-a; M_1, M_2; \beta) \), we may suppose \( a \geq 0 \). By Lemma 8.3

\[
J_q(a; M_1, M_2; \beta_1, \ldots, \beta_q) = \Gamma \int \frac{|s-a|M_2|s|^{q-1-(\beta_1+\cdots+\beta_q)}}{(1+|s-a|)^{M_1}} ds
\]

where

\[
\Gamma = \prod_{i=2}^q \int \frac{dt}{|t|^{\beta_i+\cdots+\beta_q-(q-1)}|1-t|^{\beta_{i-1}}}. \]

The conditions on \( \beta_i \)'s, \( M_1 \) and \( M_2 \) imply \( J_q(a; M_1, M_2; \beta_1, \ldots, \beta_q) < \infty \) for all \( a \). To obtain the sup on \( a > 0 \), we set \( v = s/a \). Then, denoting \( S = \sum_{i=1}^q \beta_i \), we get

\[
J_q(a; M_1, M_2; \beta) = C a^{q+M_2-S} \int_{\mathbb{R}} |v-1|^{M_2(1+a|v-1|)} |v|^{-S+(q-1)} dv, \quad (74)
\]

where \( C \) is a positive constant. We separate the integration domain in two. Suppose first that \( |v-1| \leq a^{-1} \). Then in this case we have \((1+a|v-1|)^{-M_1} \leq 1 \). Since \( |v| \) is bounded on the interval \( |v-1| < a^{-1} \) for \( a \) large then as \( a \to \infty \),

\[
\int_{|v-1| \leq a^{-1}} |v-1|^{M_2(1+a|v-1|)} |v|^{-S+(q-1)} dv = O \left( \int_{|v-1| \leq a^{-1}} |v-1|^{M_2} dv \right) = O(a^{-1-M_2}).
\]

Now suppose that \( |v-1| > a^{-1} \). Then \((1+a|v-1|)^{-M_1} \leq (a|v-1|)^{-M_1} \), and

\[
I = \int_{|v-1| > a^{-1}} |v-1|^{M_2(1+a|v-1|)} |v|^{-S+(q-1)} dv \\
\leq a^{-M_1} \int_{|v-1| > a^{-1}} |v-1|^{M_2-M_1} |v|^{-S+(q-1)} dv \\
= a^{-M_1} \left( \int_{|v| \geq 2} |v-1|^{M_2-M_1} |v|^{-S+(q-1)} dv + \int_{|v| \leq 2, |v-1| < a^{-1}} |v-1|^{M_2-M_1} |v|^{-S+(q-1)} dv \right) \\
+ a^{-M_1} \int_{|v-1| > a^{-1}} |v-1|^{M_2-M_1} |v|^{-S+(q-1)} dv.
\]

The first integral concentrates around \( v = \infty \), the second around \( v = 1 \) and the third around \( v = 0 \). The first integral is bounded, the second is

\[
O \left( \int_{|v-1| > a^{-1}} |v-1|^{M_2-M_1} dv \right) = O(a^{M_1-M_2-1}), \quad \text{as } a \to \infty,
\]

and the third is bounded. Therefore we get

\[
I = O(a^{-M_1}) + O(a^{-M_2-1}),
\]

and
since $M_2 - M_1 < -1$. Thus (74) gives
\[ J_q(a; M_1, M_2; \beta) = O(a^{-1+q-S}) \quad \text{as} \quad a \to \infty, \]
yielding the bound (73).

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