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SIMPLICIAL HOMOLOGY OF RANDOM CONFIGURATIONS

L. DECREUSEFOND, E. FERRAZ, AND H. RANDRIAMBOLOLONA

Abstract. Given a Poisson process on a d-dimensional torus, its random geometric simplicial complex is the complex whose vertices are the points of the Poisson process and simplices are given by the Čech complex associated to the coverage of each point. We compute explicitly the mean number of k-simplices as well as the mean of the Euler characteristic. Then, by means of Malliavin calculus, we show that the number of any connected geometric simplicial complex converges to the Gaussian law when the intensity of the Poisson point process tends to infinity. We use a concentration inequality to find bounds for the for the distribution of the Betti number of first order in such simplicial complex.

1. Motivation

As technology goes on [1, 2, 3], one can expect a wide expansion of the so-called *sensor networks*. Such networks represent the next evolutionary step in building, utilities, industrial, home, agriculture, defense and many other contexts [4]. These networks are built upon a multitude of small and cheap sensors which are devices with limited transmission capabilities and power. Each sensor monitors a region around itself by measuring some environmental quantities (e.g., temperature, humidity), detecting intrusion, etc, and broadcasts its collected information to other sensors or to a central node. Two questions are of paramount importance: can information be shared among the whole network, is the whole region totally monitored ?

Several researches have recently been dedicated to this problem considering a variety of situations. One can distinguish three main scenarios: those where it is possible to choose the position of each sensor, those where sensors are arbitrarily deployed in the target region with the control of a central station and those where the sensor locations are random in a decentralized system. In many cases, placing the sensors is impossible or implies a high cost. Sometimes this impossibility comes from the fact that the cost of placing each sensor is too large and sometimes the network has an inherent random behavior (like in the ad-hoc case, where users move). In addition, this policy cannot take into account the configuration of the network in the case of failure of some sensor. The drawback of the second scenario is a higher cost of sensors, since each one has to communicate with the central station. Besides, the central station itself increases the cost of the whole system. Moreover, if sensors are supposed to know their positions, an absolute positioning system has to be included in each sensor, making their hardware even more complex and then more expensive. It is thus important to investigate the third scenario: randomly located sensors, no central station. Actually, if we can predict some characteristics of the topology of a random network, the number of sensors (or, as well, the power supply of them) can be *a priori* determined such that a given network may operate with high probability. For instance, we can choose the mean number of sensors

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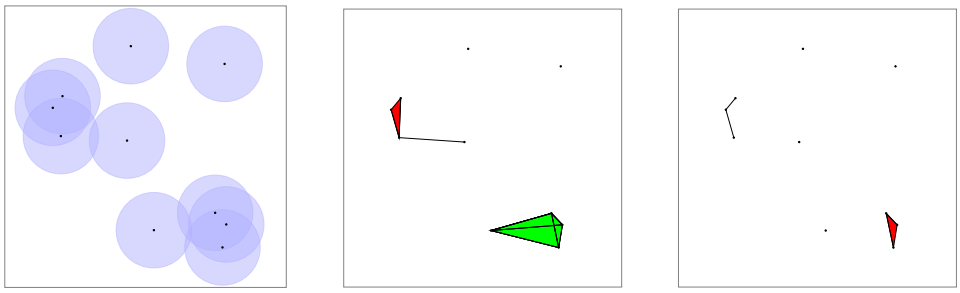


FIGURE 1. From left to right. a) Sensors and their coverage; b) simplicial complex representation when sensors are monitoring the region; c) simplicial complex representation when sensors are communicating among them.

such that, if they are randomly deployed, there is more than 99% of probability the network to be completely connected.

Usually, sensors are deployed in the plane or in the ambient space, thus mathematically speaking, one has to deal with configurations in R^2, R^3 or a manifold. The recent works of Ghrist and his collaborators [5, 6] show how, in any dimension, algebraic topology can be used to compute the coverage of a given configuration of sensors. Trying to pursue their work for random settings, we completely solved the problem in one dimension in [7] by basic methods inspired by queuing theory, without using the forthcoming sophisticated tools of algebraic topology. Since we cannot order points in R^d, it is not possible to generalize the results obtained in this earlier work to higher dimension.

A simplicial complex is a generalization of a graph: while we represent a graph with points and edges, a simplicial complex can be represented by points, edges, filled triangles, filled tetrahedrons and so on. It is usual to intepret ϵ as the radius of monitoring but a different interpretation can be given if the sensors are communicating among them. In this case, we suppose that sensors have a power supply allowing them to transmit theirs ID's and, at the same time, sensors have receivers which can identify the transmitted ID's of other sensors above a threshold power. The sensors, knowing mutually the ID's of the close neighbors, are considered connected, creating an information network. The problem remains analogous as the previous one, except that we substitute the coverage radius ϵ by a communication one of $\epsilon/2$. We can see examples of simplicial complexes representations given by sensors communicating among them or monitoring a region in Fig. 1.

In this work we consider that sensors are the atoms of a Poisson point process. Instead of the Euclidean norm, we use the maximum norm along this paper. We consider this for three reasons: this norm represents a superior and an inferior limits for the euclidean norm (we can inscribe and circumscribe a circle with two squares); due to the random interactions with the environment (causing shadowing and fading), even the euclidean norm cannot capture with precision the real behavior of this kind of sensor networks, so we choose the norm that allows us to simplify the calculations; using the maximum norm, the Čech complex becomes equal to the Rips-Vietoris complexes, which allows us to apply directly some results of the algebraic topology. Finally, we assume that sensors lie over d-torus with sides a, $\mathbb{T}^d_a$. This choice was motivated by three factors: it avoids the side effects, it helps to determine weather or not a sensor network in the d-box is completely covered

and if ϵ is small compared to a, the calculations for all parameters in the d-torus are a good approximation for the $[0, a]^d$ box.

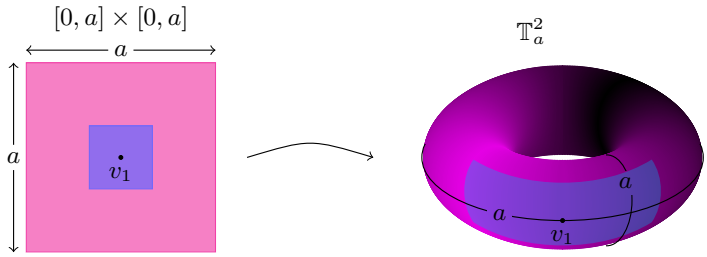


FIGURE 2. Illustration of the coverage of a point and the region where points can lie, in the 2 dimensional case.

One of the main results of this paper is the explicit expression for the mean number of k-simplices in such complex and the mean of the Euler characteristic. To find those means, we start from the topological tools described in [5] followed by the application of the Campbell's measure. This result allows us to apply a concentration inequality to analyze the connectivity of the network. The other result is the convergence in law of the number of connected simplicial complex. As consequence of a theorem in [8] obtained by the means of the Malliavin calculus, we show that the number of connected simplices tends, in law, to a Gaussian, when the intensity measure of the process tends to infinity.

The paper is organized in the following way: Section 2 is a primer on algebraic topology and Malliavin calculus. In Section 3, the average number of simplices and the mean of the Euler characteristics are computed. Section 4 is devoted to asymptotic analysis: We prove a central limit theorem valid for any finite simplex and we use concentration inequality to estimate the distribution queue of the number of simplices of any order.

2. Algebraic Topology

For further reading on topology, see [9, 10, 11]. Graphs can be generalized to more generic topological objects known as simplicial complexes. While graphs model binary relations, simplicial complexes represent higher order relations. Given a set of points V, a k-simplex is an unordered subset $\{v_0, v_1, \cdots, v_k\}$ where $v_i \in V$ and $v_i \neq v_j$ for all $i \neq j$. The faces of the k-simplex $\{v_0, v_1, \cdots, v_k\}$ are defined as all the $(k-1)$-simplices of the form $\{v_0, \cdots, v_{j-1}, v_{j+1}, \cdots, v_k\}$ with $0 \leq j \leq k$. A simplicial complex is a collection of simplices which is closed with respect to the inclusion of faces, i.e., if $\{v_0, v_1, \cdots, v_k\}$ is a k-simplex then all its faces are in the set of $(k-1)$-simplices.

Given $\mathcal{U} = (U_v, v \in \mathfrak{T})$ a collection of open sets, the Čech complex of $\mathcal{U}$ denoted by $\mathcal{C}(\mathcal{U})$, is the abstract simplicial complex whose k-simplices correspond to $(k+1)$-tuples of distinct elements of $\mathcal{U}$ that have non empty intersection, so $\{v_0, v_1, \cdots, v_k\}$ is a k-simplex if and only if $\bigcap_{i=0}^{k} U_{v_k} \neq \emptyset$.

One can define an orientation for a simplicial complex by defining an order on vertices. A change in the orientation corresponds to a change in the sign of the

coefficient as

$$[v_0, \cdots, v_i, \cdots, v_j, \cdots, v_k] = -[v_0, \cdots, v_j, \cdots, v_i, \cdots, v_k].$$

For each integer k, $C_k(X)$ is the vector space spanned by the set of oriented k-simplices of X. The boundary map ∂_k is defined to be the linear transformations $\partial_k : C_k \to C_{k-1}$ which acts on basis elements $[v_0, \cdots, v_k]$ via

$$\partial_k[v_0, \cdots, v_k] = \sum_{i=0}^{k} (-1)^k [v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_k]. \tag{1}$$

Examples of such operations are given in Fig. 3.

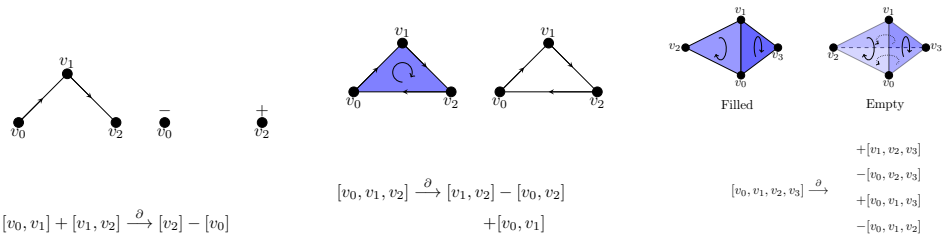


FIGURE 3. Examples of boundary maps. From left to right. An application over 1-simplices. Over a 2-simplex. Over a 3-simplex, turning a filled tetrahedron to an empty one.

This map gives rise to a chain complex: a sequence of vector spaces and linear transformations

$$\cdots \xrightarrow{\partial_{k+2}} C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X).$$

A standard result then asserts that for any integer k,

$$\partial_k \circ \partial_{k+1} = 0.$$

If one defines

$$Z_k = \ker \partial_k \text{ and } B_k = \mathrm{im}\partial_{k+1},$$

this induces that $B_k \subset Z_k$.

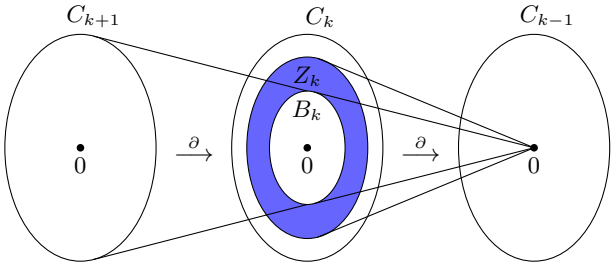


FIGURE 4. A chain complex showing the sets C_k, Z_k and B_k.

The k-dimensional homology of X, denoted $H_k(X)$ is the quotient vector space,

$$H_k(X) = \frac{Z_k(X)}{B_k(X)}. \tag{2}$$

and the k-th Betti number of X is its dimension:

$$\beta_k = \dim H_k = \dim Z_k - \dim B_k.$$

Let s_k be the number of k-simplices in a simplicial complex X. The well known topological invariant named Euler characteristic for X, denoted by $\chi(X)$, is an integer defined by:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \beta_i. \tag{3}$$

A well known theorem states that this is also given by:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i s_i. \tag{4}$$

The simplicial complexes we consider are of a special type. They can be considered as a generalization of geometric random graphs.

Definition 1. *Given ω a finite set of points on the torus. For $\epsilon > 0$, we define $\mathcal{U}_\epsilon(\omega) = \{B_{d_\infty}(v, \epsilon), v \in \omega\}$ and $\mathcal{C}_\epsilon(\omega) = \mathcal{C}(\mathcal{U}_\epsilon(\omega))$, where $B_{d_\infty}(x, r) = \{y \in \mathbb{T}^a_d, \|x - y\|_\infty < r\}$.*

Theorem 1. *Suppose $\epsilon < a/4$. Then $\mathcal{C}_\epsilon(\omega)$ has the same homotopy type as $\mathcal{U}_\epsilon(\omega)$. In particular they have the same Betti numbers.*

Proof. This will follow from the so-called nerve lemma of Leray, as stated in [12, Theorem 7.26] or [13, Theorem 10.7]. One only needs to check that any non-empty intersection of sets $B_{d_\infty}(v, \epsilon)$ is contractible.

Consider such a non-empty intersection, and let x be a point contained in it. Then, since $\epsilon < a/4$, the ball $B_{d_\infty}(x, 4\epsilon)$ can be identified with a cube in the Euclidean space. Then each $B_{d_\infty}(v, \epsilon)$ containing x is contained in $B_{d_\infty}(x, 4\epsilon)$, hence also becomes a cube with this identification, hence convex. Then the intersection of these convex sets is convex, hence contractible. □

Definition 2. *Let ω be a finite set of points in $\mathbb{T}^a_d$. For any $\epsilon > 0$, the Rips-Vietoris complex of ω, $\mathcal{R}_\epsilon(\omega)$, is the abstract complex whose k-simplices correspond to unordered $(k+1)$-tuples of points in ω which are pairwise within distance less than ϵ of each other.*

Lemma 2. *For the torus $\mathbb{T}^d_a$ equipped with the product distance d_∞, $\mathcal{R}_\epsilon(\omega)$ has the homotopy type of the Čech complex $\mathcal{C}_{2\epsilon}(\omega)$*

The proof is given in [5] in a slightly different context, but it is easy to check that it works here as well. It must be pointed out that Čech and Rips-Vietoris simplicial complexes can be defined similarly for any distance on $\mathbb{T}^d_a$ but it is only for the product distance that the homotopy type of both complexes coincides.

Proposition 3. *Let $\omega \in \mathbb{T}^d_a$ be a set of points, generating the simplicial complex $\mathcal{C}_\epsilon(\omega)$. Then, if $i > d$, $\beta_i(\omega) = 0$.*

Proof. By Theorem 1, $\mathcal{C}_\epsilon(\omega)$ has the same homology as $\mathcal{U}_\epsilon(\omega)$. But $\mathcal{U}_\epsilon(\omega)$ is an open manifold of dimension d, so its Betti numbers $\beta_i(\omega)$ vanish for $i > d$, see for example [14, Theorem 22.24]. □

Proposition 4. *Let $\omega \in \mathbb{T}^d_a$ be a set of points, generating the simplicial complex $\mathcal{C}_\epsilon(\omega)$. There are only two possible values for the d-th Betti number of $\mathcal{C}_\epsilon(\omega)$:*

i) $\beta_d = 0$, or
ii) $\beta_d = 1$.

If the second holds, then we also have $\chi(\mathcal{C}_\epsilon(\omega)) = 0$.

Proof. By Theorem 1, $\mathcal{C}_\epsilon(\omega)$ has the same homology as $\mathcal{U}_\epsilon(\omega)$. Now, $\mathcal{U}_\epsilon(\omega)$ is an open submanifold of the torus, so there are only two possibilities:

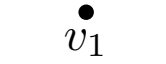
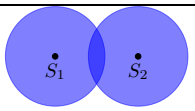
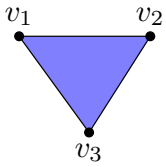
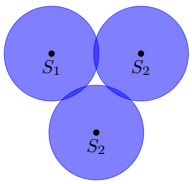
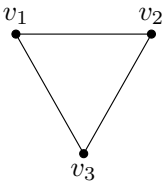
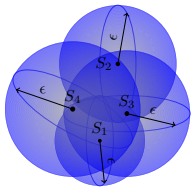
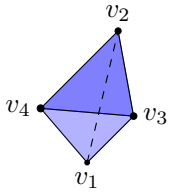
Sensor network coverage	Čech complex representation
	
	
	
	

TABLE 1. Topological representation of the coverage of a sensor network. Each node v represents a sensor. From top to bottom, the highest order simplex is a vertex, an edge, a triangle, three edges, a tetrahedron.

i) $\mathcal{U}_\epsilon(\omega)$ is a strict open submanifold, hence non-compact
ii) $\mathcal{U}_\epsilon(\omega) = \mathbb{T}^d_a$.

In the first case, $\beta_d(\omega) = 0$ by [14, Corollary 22.25]. In the second case $\mathcal{C}_\epsilon(\omega)$ has same homology as the torus, hence $\beta_d(\omega) = 1$ and $\chi(\omega) = 0$. □

2.1. **Application to sensor networks.** We now interpret the topological properties of simplicial complexes in terms of connectivity and coverage. In terms of coverage in a network, a 0-simplex represents a single sensor and the existence of a k-simplex means that the $(k+1)$ points of this simplex are covering the convex hull containing those points. We can see in Figure 1, examples of some simplices and their interpretation in terms of sensor networks.

In a very intuitive fashion, the number of k-simplices itself shows some tendency in the network: if in two networks with identical number of sensors, one of them has more 1-simplices than the other, this first one has a tendency to be more connected; by the same reason, if a network has more 2-simplices than another one, the region on the first case tends to be more strongly covered.

In a more sophisticated way, Theorem 1 formalizes that, in order to determine coverage of sensors, it suffices interpret them as Čech complexes. Unfortunately, a moment of thought shows that constructing the Čech complex cannot be done by pairwise only communications between sensors. Thus, the only complex that can be computed this way is the Rips-Vietoris complex.

An interpretation to Euler characteristic is given by Proposition 4, where we see that $\chi = 0$ is a necessary condition to have a complete coverage of the torus, and $\beta_d = 1$ is a necessary and sufficient condition. This could in turn translate into conditions for coverage in $[0, a]^d$ when considered as embedded in Euclidean space (i.e., not as a torus), but then one needs to be careful about border effects. For example, one can say that $\beta_d = 1$ is a sufficient condition for coverage of $[\epsilon, a - \epsilon]^d$.

3. Stochastic Model

3.1. **Poisson point process.** To characterize the randomness of the system, we consider that the set of points is represented by a Poisson point process ω with intensity λ in a Polish space Y. The space of configurations on Y, is the set of locally finite simple point measures (cf [15]):

$$\Omega^Y = \left\{ \omega = \sum_{k=0}^{n} \delta(x_k) \ : \ (x_k)_{k=0}^{k=n} \subset Y, \ n \in \mathbb{N} \cup \{\infty\} \right\},$$

where $\delta(x)$ denotes the Dirac measure at $x \in Y$. Simple measure means that $\omega(\{x\}) \leq 1$ for any $x \in Y$. Locally finite means that $\omega(K) < \infty$ for any compact K of Y. It is often convenient to identify an element ω of Ω^Y with the set corresponding to its support, i.e., $\sum_{k=0}^{n} \delta(x_k)$ is identified with the unordered set $\{x_1, \cdots, x_n\}$. For $A \in \mathcal{B}(Y)$, we have $\delta(x_k)(A) = \mathbf{1}_{[x_k \in A]}$, so

$$\omega(A) = \sum_{x_k \in \omega} \mathbf{1}_{[x_k \in A]} = \int_A \mathrm{d}\omega(x),$$

counts the number of atoms in A. The configuration space Ω^Y is endowed with the vague topology and its associated σ-algebra denoted by $\mathcal{F}^Y$. Since ω is a Poisson point process of intensity λ:

i) For any A, $\omega(A)$ is a random variable of parameter $\lambda S(A)$, i.e.,

$$\mathbf{P}(\omega(A) = k) = e^{-\lambda(A)} \frac{(\lambda(A))^k}{k!}.$$

ii) For $A' \in \mathcal{B}(Y)$, for any disjoints A, A', the random variables $\omega(A)$ and $\omega(A')$ are independent.

Define $\Delta_n = \{(x_1, \cdots, x_n) \in Y^n \mid x_i \neq x_j, \forall i \neq j\}$. Let $f(x_1, \cdots, x_n)$ be a measurable function and let $F(\omega)$ be a random variable given by

$$F(\omega) = \sum_{\substack{x_i \in \omega \cap A, 1 \leq i \leq n \\ x_i \neq x_j \ if \ i \neq j}} f(x_1, \cdots, x_n) = \int_{A \cap \Delta_n} f(x_1, \cdots, x_n) \ \mathrm{d}\omega(x_1) \cdots \ \mathrm{d}\omega(x_n),$$

A well known property of the Poisson point processes [16] states that

$$\mathbf{E}_\lambda \left[F(\omega) \right] = \int_A f(x_1, \cdots, x_n) \ \mathrm{d}\lambda(x_1) \cdots \ \mathrm{d}\lambda(x_n).$$

A real function $f \ : \ Y^n \to \mathbb{R}$ is called symmetric if

$$f(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = f(x_1, \cdots, x_n)$$

for all permutations σ of $\mathfrak{S}_n$.The space of symmetric square integrable random variables is denoted by $L^2(\lambda)^{\circ n}$. For $f \in L^2(\lambda)^{\circ n}$, the multiple Poisson stochastic integral $I_n(f_n)$ is then defined as

$$I_n(f_n)(\omega) = \int_{\Delta^n} f_n(x_1, \cdots, x_n)(\ \mathrm{d}\omega(x_1) - \ \mathrm{d}\lambda(x_1)) \cdots (\ \mathrm{d}\omega(x_n) - \ \mathrm{d}\lambda(x_n)).$$

If $f_n \in L^2(\lambda)^{\circ n}$ and $g_m \in L^2(\lambda)^{\circ m}$, the isometry formula

$$\mathbf{E}_\lambda \left[I_n(f_n) I_m(g_m) \right] = n! \mathbf{1}_{[m=n]} \langle f_n, g_m \rangle_{L^2(\lambda)^{\circ n}} \tag{5}$$

holds true (see [15]). Furthermore, we have:

Theorem 5. *Every random variable $F \in L^2(\Omega^Y, \mathbf{P})$ admits a (unique) Wiener-Poisson decomposition of the type*

$$F = \mathbf{E}_\lambda \left[F \right] + \sum_{n=1}^{\infty} I_n(f_n),$$

where the series converges in $L^2(\mathbf{P})$ and, for each $n \geq 1$, the kernel f_n is an element of $L^2(\lambda)^{\circ n}$. Moreover, we have the isometry

$$\| F - \mathbf{E}_\lambda \left[F \right] \|^2_{L^2(\lambda)^{\circ n}} = \sum_{n=1}^{\infty} n! \| f_n \|^2_{L^2(\mathbb{R}_+)^{\circ n}}. \tag{6}$$

For $f_n \in L^2(\lambda)^{\circ n}$ and $g_m \in L^2(\lambda)^{\circ m}$, we define $f_n \otimes^l_k g_m$, $o \leq l \leq k$, to be the function:

$$(y_{l+1}, \cdots, y_n, x_{k+1}, \cdots, x_m) \longmapsto \int_{Y^l} f_n(y_1, \cdots, y_n) g_m(y_1, \cdots, y_k, x_{k+1}, \cdots, x_m)\ \mathrm{d}\lambda(y_1) \ldots\ \mathrm{d}\lambda(y_l). \tag{7}$$

We denote by $f_n \circ^l_k g_m$ the symmetrization in $n + m - k - l$ variables of $f_n \otimes^l_k g_m$, $0 \leq l \leq k$. This leads us to the next proposition, shown in [15]:

Proposition 6. *For $f_n \in L^2(\lambda)^{\circ n}$ and $g_m \in L^2(\lambda)^{\circ m}$, we have*

$$I_n(f_n) I_m(g_m) = \sum_{s=0}^{2(n \wedge m)} I_{n+m-s}(h_{n,m,s}),$$

where

$$h_{n,m,s} = \sum_{s \leq 2i \leq 2(s \wedge n \wedge m)} i! \binom{n}{i} \binom{m}{i} \binom{i}{s-i} f_n \circ^{s-i}_i g_m$$

belong to $L^2(\lambda)^{\circ n+m-s}$, $0 \leq s \leq 2(m \wedge n)$.

In what follows, given $f \in L^2(\lambda)^{\circ q}$ $(q \geq 2)$ and $t \in Y$, we denote by $f(*, t)$ the function on Y^{q-1} given by $(x_1, \cdots, x_{q-1}) \longmapsto f(x_1, \cdots, x_{q-1}, t)$.

Definition 3. *Let* $\mathrm{Dom}\, D$ *be the the set of random variables $F \in L^2(P)$ admitting a chaotic decomposition such that*

$$\sum_{n=1}^{\infty} q q! \| f_n \|^2 < \infty.$$

Let D be defined by

$$D \ : \ \mathrm{Dom}\, D \to L^2(\Omega^Y \times Y, P \times \lambda)$$
$$F = \mathbf{E}_\lambda \left[F \right] + \sum_{n \geq 1} I_n(f_n) \longmapsto D_t F = \sum_{n \geq 1} n I_{n-1}(f_n(*, t)).$$

It is known, cf. [17]*, that we also have*

$$D_t F(\omega) = F(\omega \cup \{t\}) - F(\omega),\ dP \times dt\ a.e.. \tag{8}$$

Definition 4. *The Ornstein-Uhlenbeck operator L is given by*

$$LF = -\sum_{n=1}^{\infty} nI_n(f_n),$$

whenever $F \in \operatorname{Dom} L$, given by those $F \in L^2 P$ such that their chaotic expansion verifies

$$\sum_{n=1}^{\infty} q^2 q! \|f_n\|^2 < \infty.$$

Note that $\mathbf{E}_\lambda [LF] = 0$, by definition and ((5)).

Definition 5. *For $F \in L^2(\mathbf{P})$ such that $\mathbf{E}_\lambda [F] = 0$, we may define L^{-1} by*

$$L^{-1} F = -\sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n).$$

Combining Stein's method and Malliavin calculus yields the following theorem, see [8]:

Theorem 7. *Let $F \in \operatorname{Dom} D$ be such that $\mathbf{E}_\lambda [F] = 0$ and $Var(F) = 1$. Then,*

$$d_W(F, \mathcal{N}(0,1)) \leq \mathbf{E}_\lambda \left[\left| 1 - \int_Y [D_t F \times D_t L^{-1} F]\ d\lambda(t) \right| \right] + \int_Y \mathbf{E}_\lambda \left[|D_t F|^2 \left| D_t L^{-1} F \right| \right]\ d\lambda(t). \tag{9}$$

Another result from the Malliavin calculus used in this work is the following one, quoted from [15]:

Theorem 8. *Let $F \in \operatorname{Dom} D$ be such that $DF \leq K$, a.s., for some $K \geq 0$ and $\|DF\|_{L^\infty(\Omega, L^2(Y))} < \infty$. Then*

$$\mathbf{P}(F - \mathbf{E}_\lambda [F] \geq x) \leq \exp\left(-\frac{x}{2K} \log\left(1 + \frac{xK}{\|DF\|_{L^\infty(\Omega, L^2(Y))}} \right) \right). \tag{10}$$

4. Mean value of Euler characteristic

Consider that a Poisson point process ω generates a Čech complex $\mathcal{C}_\epsilon(\omega)$, and, even though the number of k-simplices, the Betti's number and the Euler characteristic are functions of $\mathcal{C}_\epsilon(\omega)$, we denote them, respectively, $N_k(\mathcal{C}_\epsilon(\omega)) = N_k$, $\beta_k(\mathcal{C}_\epsilon(\omega)) = \beta_k$ and $\chi(\mathcal{C}_\epsilon(\omega)) = \chi$. In this section, we evaluate the mean of the number of k-simplices, $\mathbf{E}_\lambda [N_k]$ and the mean Euler characteristic, $\mathbf{E}_\lambda [\chi]$.

Theorem 9. *Let $\epsilon \leq a/6$. Then, the mean number of k-simplices $N_k(\mathcal{C}_\epsilon(\omega))$ is given by*

$$\mathbf{E}_\lambda [N_k] = \frac{\lambda^{k+1} (a(k+1)(2\epsilon)^k)^d}{(k+1)!}. \tag{11}$$

Proof. By Lemma 2, $k+1$ points are forming a k-simplex whenever they are two-by-two closer than 2ϵ from each other. Let us define $h(v_0, ..., v_k)$ as

$$h(v_0, ..., v_k) = \prod_{0 \leq i < j \leq k} \mathbf{1}_{[\|v_i - v_j\| < 2\epsilon]},$$

which determines if a set of $k+1$ distinct ordered points generates a k-simplex. Moreover, if $(u_{i,1},...,u_{i,d})$ represents the coordinates of a point v_i, we can separate the indicator function as follows:

$$\mathbf{1}_{[\|v_i-v_j\|<2\epsilon]} = \prod_{l=0}^{d} \mathbf{1}_{[\{|u_{i,l}-u_{j,l}|<2\epsilon\}\cup\{|u_{i,l}-u_{j,l}|>a-2\epsilon\}]}.$$

The number of k-simplices can be counted by the expression:

$$N_k = \frac{1}{(k+1)!} \sum_{\substack{v_0,\dots,v_k\in\omega \\ v_i\neq v_j \ if \ i\neq j}} h(v_0,\cdots,v_k).$$

Since ω is a Poisson point process of intensity λ, for a borel sets A_i, i integer we have:

$$\mathbf{E}_\lambda\left[\sum_{\substack{v_0,\dots v_k\in\omega \\ v_i\neq v_j \ if \ i\neq j}} f(v_0,...,v_k)\right] = \lambda^{k+1}\int_{A_0}\cdots\int_{A_k} f(v_0,...,v_k)\ \mathrm{d}v_0\dots\ \mathrm{d}v_k.$$

Taking $\frac{1}{(k+1)!}h = f$, $A_i = \mathbb{T}_a^d$ and defining

$$\{|x_i-x_j|<2\epsilon\}\cup\{|x_i-x_j|>a-2\epsilon\} = d_\epsilon(x_i,x_j),$$

we have:

$$\begin{aligned}
\mathbf{E}_\lambda[N_k] &= \frac{\lambda^{k+1}}{(k+1)!}\int_{\mathbb{T}_a^d}\cdots\int_{\mathbb{T}_a^d} h(v_0,...,v_k)\ \mathrm{d}v_0\dots\ \mathrm{d}v_k \\
&= \frac{\lambda^{k+1}}{(k+1)!}\prod_{l=0}^{k}\int_0^a\cdots\int_0^a \prod_{0\le i<j\le k}\mathbf{1}_{[d_\epsilon(u_{i,l},u_{j,l})]}\ \mathrm{d}u_{0,l}\dots\ \mathrm{d}u_{k,l} \\
&= \frac{\lambda^{k+1}}{(k+1)!}\left(\int_0^a\cdots\int_0^a \prod_{0\le i<j\le k}\mathbf{1}_{[d_\epsilon(x_i,x_j)]}\ \mathrm{d}x_0\dots\ \mathrm{d}x_k\right)^d \qquad (12) \\
&= \frac{\lambda^{k+1}}{(k+1)!}\left(\int_0^a \underbrace{\int_0^a \mathbf{1}_{[d_\epsilon(x_k,x_{k-1})]}\cdots\int_0^a}_{m\ \text{integrals}} \prod_{i=k-m+1}^{k}\mathbf{1}_{[d_\epsilon(x_i,x_{k-m})]}\cdots\right. \\
&\qquad\qquad \left.\int_0^a\prod_{i=1}^{k}\mathbf{1}_{[d_\epsilon(x_i,x_0)]}\ \mathrm{d}x_0\dots\ \mathrm{d}x_k\right)^d. \qquad (13)
\end{aligned}$$

Since $6\epsilon\le a$, the integration region is convex (see Fig. 5).

Then, we can rewrite the integral in Eq. (13) as

$$\begin{aligned}
&\int_0^a\int_0^a \mathbf{1}_{[d_\epsilon(x_k,x_{k-1})]}\cdots\int_0^a\prod_{i=1}^{k}\mathbf{1}_{[d_\epsilon(x_i,x_0)]}\ \mathrm{d}x_0\dots\ \mathrm{d}x_k = \qquad (14) \\
&\qquad\int_0^a\int_{x_k-2\epsilon}^{x_k+2\epsilon}\int_{\max(x_k,x_{k-1})-2\epsilon}^{\min(x_k,x_{k-1})+2\epsilon}\cdots\int_{\max(x_k,\cdots,x_1)-2\epsilon}^{\min(x_k,\cdots,x_1)+2\epsilon}\mathrm{d}x_0\dots\ \mathrm{d}x_k.
\end{aligned}$$

Then, consider a subset of the integration region $[0,a]^d$ of Eq. (14), defined as $A_{0,1,\dots,k}$, such that $x_0\ge x_1\ge\ldots\ge x_k$. In this case, we can write the integral over

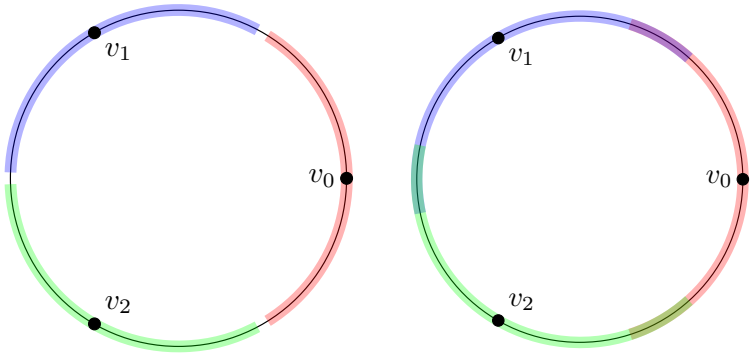


FIGURE 5. a) Maximum cover in $\mathbb{T}_a$ and $\epsilon = a/6$. The red region shows the cover of a point v_0, the blue region is the cover of v_1 and the green region is the cover of v_2. b) Maximum cover in the same conditions of a) when $\epsilon = a/5$. In this case, we the three covers intersect each other pairwise, but there is no intersection of the three covers.

$A_{0,\,1,\,\cdots,\,k}$ as:

$$\int_0^a \int_{x_k-2\epsilon}^{x_k+2\epsilon} \cdots \int_{\max(x_k,\cdots,x_1)-2\epsilon}^{\min(x_k,\cdots,x_1)+2\epsilon} \mathbf{1}_{[x_i \geq x_j \text{ if } i\leq j]}\, \mathrm{d}x_0 \ldots\, \mathrm{d}x_k = \int_0^a \int_{x_k}^{x_k+2\epsilon} \int_{x_{k-1}}^{x_k+2\epsilon} \cdots \int_{x_1}^{x_k+2\epsilon} \mathrm{d}x_0 \ldots\, \mathrm{d}x_k. \tag{15}$$

For $\sigma \in \mathfrak{S}_k$, we denote by A_σ the set $A_{\sigma(1),\,\cdots,\,\sigma(k)}$. Then,

$$\bigcup_{\sigma\in\mathfrak{S}_k} A_\sigma = [0,a]^d.$$

Moreover, since the function $h(x_0, \cdots, x_k)$ is symmetric, we can exchange the integration variables in the integral of Eq. (12) without changing its result. As a consequence, if $\sigma \in \mathfrak{S}_k$,

$$\int_{A_\sigma} \cdots \int \prod_{0\leq i<j\leq k} \mathbf{1}_{[d_\epsilon(x_i,\,x_j)]}\, \mathrm{d}x_0 \ldots\, \mathrm{d}x_k = \int_{A_\sigma} \cdots \int \prod_{0\leq i<j\leq k} \mathbf{1}_{[d_\epsilon(x_i,\,x_j)]}\, \mathrm{d}x_{i_0} \ldots\, \mathrm{d}x_{i_k}.$$

Thus, we have

$$\begin{aligned}
&\int_0^a \cdots \int_0^a \prod_{0\leq i<j\leq k} \mathbf{1}_{[d_\epsilon(x_i,\,x_j)]}\, \mathrm{d}x_0 \ldots\, \mathrm{d}x_k \\
&\qquad = \sum_{\sigma\in\mathfrak{S}_k)} \int_{A_\sigma} \cdots \int \prod_{0\leq i<j\leq k} \mathbf{1}_{[d_\epsilon(x_i,\,x_j)]}\, \mathrm{d}x_{i_0} \ldots\, \mathrm{d}x_{i_k} \\
&\qquad = (k+1)! \int_{A_{\mathrm{Id}}} \cdots \int \prod_{0\leq i<j\leq k} \mathbf{1}_{[d_\epsilon(x_i,\,x_j)]}\, \mathrm{d}x_0 \ldots\, \mathrm{d}x_k \\
&\qquad = (k+1)! \int_0^a \int_{x_k}^{x_k+2\epsilon} \int_{x_{k-1}}^{x_k+2\epsilon} \cdots \int_{x_1}^{x_k+2\epsilon} \mathrm{d}x_0 \ldots\, \mathrm{d}x_k.
\end{aligned}$$

Then, by the change of variables $y_{k-i} = (x_{k-i} - x_k)/2\epsilon$ for $i = \overline{1,k}$, we get:

$$(k+1)! \int_0^a \int_{x_k}^{x_k+2\epsilon} \int_{x_{k-1}}^{x_k+2\epsilon} \dots \int_{x_1}^{x_k+2\epsilon} \mathrm{d}x_0 \dots \mathrm{d}x_k = (2\epsilon)^k (k+1)! \int_0^a \int_0^1 \int_{y_{k-1}}^1 \dots \int_{y_1}^1 \mathrm{d}y_0 \dots \mathrm{d}y_{k-1}\, \mathrm{d}x_k.$$

The integral in the right-handed term is evaluated substituting all y_i for $y_i' - 1$, so

$$(2\epsilon)^k (k+1)! \int_0^a \int_{-1}^0 \int_{y_{k-1}'}^0 \dots \int_{y_1'}^0 \mathrm{d}y_0' \dots \mathrm{d}y_{k-1}'\, \mathrm{d}x_k = (2\epsilon)^k (k+1)! a \frac{1}{k!} = a(2\epsilon)^k (k+1). \tag{16}$$

Finally, plug Eq. (16) into Eq. (13) to obtain:

$$\mathbf{E}_\lambda [N_k] = \frac{\lambda^{k+1} (a(k+1)(2\epsilon)^k)^d}{(k+1)!},$$

and thus the proof is complete. □

Remark 1. *The possibility of writing $\mathbf{E}_\lambda [N_k]$ as Eq. (12) is due the fact that we use the maximum norm. This simplifies the calculations since we can treat each component individually. However, considering the Euclidean norm it is still possible to find a closed-form expression for $\mathbf{E}_\lambda [N_1]$ and $\mathbf{E}_\lambda [N_2]$ when we consider the Rips-Vietoris complex in $\mathbb{T}_a^2$. These expressions, obtained after solving some integrals, are given by:*

$$\begin{aligned} \mathbf{E}_\lambda [N_1] &= \frac{\pi (a\lambda\epsilon)^2}{2}, \\ \mathbf{E}_\lambda [N_2] &= \pi \left(\pi - \frac{3\sqrt{3}}{4} \right) \frac{\lambda^3 a^2 \epsilon^4}{6}. \end{aligned}$$

Consider now the Bell's polynomial $B_d(x)$, defined as (see [18])

$$B_n(x) = \sum_{k=0}^n \left\{ {n \atop k} \right\} x^k,$$

where n is an positive integer and $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind. An equivalent definition of B_n can be:

$$B_n(x) = e^{-x} \sum_{k=0}^\infty \frac{x^k k^d}{k!}. \tag{17}$$

Theorem 10. *Let $\epsilon \leq a/6$. The mean of the Euler characteristic mean of the simplicial complex $\mathcal{C}_\epsilon(\omega)$ is given by*

$$\mathbf{E}_\lambda [\chi] = \left(\frac{a}{2\epsilon} \right)^d e^{-\lambda (2\epsilon)^d} (-B_d(-\lambda (2\epsilon)^d)). \tag{18}$$

Proof. Since

$$N_k \leq \frac{1}{k!} \prod_{j=0}^{N_0 - 1} (N_0 - j) \leq \frac{N_0^k}{k!},$$

then

$$\sum_{k=0}^\infty N_k \leq \sum_{k=0}^\infty \frac{N_0^k}{k!} = e^{N_0}.$$

As $\mathbf{E}_\lambda\left[e^{N_0}\right] < \infty$, we have $\mathbf{E}_\lambda\left[\sum_{k=0}^{\infty}(-1)^k N_k\right] = \sum_{k=0}^{\infty}(-1)^k \mathbf{E}_\lambda\left[N_k\right]$ and

$$\begin{aligned}
\mathbf{E}_\lambda\left[\chi\right] &= \mathbf{E}_\lambda\left[\sum_{k=0}^{\infty}(-1)^k N_k\right] \\
&= \sum_{k=0}^{\infty}(-1)^k \mathbf{E}_\lambda\left[N_k\right] \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{k+1}(a(k+1)(2\epsilon)^k)^d}{(k+1)!} \\
&= \frac{a^d}{-(2\epsilon)^d}\sum_{k=0}^{\infty}\frac{(-\lambda(2\epsilon)^d)^{k+1}(k+1)^d}{(k+1)!} \\
&= \frac{a^d e^{-\lambda(2\epsilon)^d}}{-(2\epsilon)^d} e^{\lambda(2\epsilon)^d}\sum_{k=0}^{\infty}\frac{(-\lambda(2\epsilon)^d)^k k^d}{k!} \\
&= \left(\frac{a}{2\epsilon}\right)^d e^{-\lambda(2\epsilon)^d}(-B_d(-\lambda(2\epsilon)^d)).
\end{aligned}$$

The proof is thus complete. □

If we take $d = 1$, $d = 2$ and $d = 3$, we obtain:

$$\begin{aligned}
\mathbf{E}_\lambda\left[\chi\right]\left(\mathcal{C}_\epsilon(\omega)\right)_{\omega\in\mathbb{T}_{[a]}} &= a\lambda e^{-\lambda 2\epsilon}, \\
\mathbf{E}_\lambda\left[\chi\right]\left(\mathcal{C}_\epsilon(\omega)\right)_{\omega\in\mathbb{T}^2_{[a^2]}} &= a^2\lambda e^{-\lambda(2\epsilon)^2}\left(1-\lambda(2\epsilon)^2\right), \\
\mathbf{E}_\lambda\left[\chi\right]\left(\mathcal{C}_\epsilon(\omega)\right)_{\omega\in\mathbb{T}^3_{[a^3]}} &= a^3\lambda e^{-\lambda(2\epsilon)^3}\left(1-3\lambda(2\epsilon)^3+(\lambda(2\epsilon)^3)^2\right).
\end{aligned}$$

Remark 2. *For c a positive real, $\mathbf{E}_\lambda\left[\chi\right]$ is invariant under the transformation $\lambda' = \lambda/c$, $\epsilon' = c\epsilon$ and $a' = ca$. Taking $c = 1/2\epsilon$, we obtain:*

$$\mathbf{E}_\lambda\left[\chi\right] = a'^d e^{-\lambda'}(-B_d(-\lambda')). \tag{19}$$

Hence, the mean depends actually only on a' and λ'.

The following result is well known.

Lemma 11. *If $B_d(x)$ is the Bell's polynomial and for $d \geq 1$, the following relations are valid:*

$$\begin{aligned}
\frac{d}{dx}B_d(x) &= \frac{B_{d+1}(x)}{x} - B_d(x), \\
\frac{d}{dx}(e^x B_d(x)) &= \frac{e^x}{x}B_{d+1}(x).
\end{aligned}$$

According to these relations, it is routine to prove the following theorem.

Theorem 12. *The function $(\lambda \mapsto \mathbf{E}_\lambda\left[\chi\right])$ has exactly d non-negative real roots. Moreover, between each consecutive roots and after the last one, there is exactly one critical point.*

We can see by the expression of $\mathbf{E}_\lambda\left[\chi\right]$ that this quantity tends to 0 as λ tends to infinity. This convergence is due the fact that the Euler characteristic of the Čech complex of the cover tends to the Euler characteristic of the d-Torus where the points are deployed. This is shown in the following theorem.

Theorem 13. *The Betti numbers of $\mathcal{C}(\mathcal{U}_\epsilon)$ converge in probability to the Betti number of the torus as λ goes to infinity:*

$$\mathbf{P}_\lambda\left(\bigcap_{i=0}^{d}\left(\beta_i(\mathcal{C}_\epsilon) = \beta_i(\mathbb{T}^d_{[a]})\right)\right) \xrightarrow{\lambda\to\infty} 1.$$

Proof. Let $\eta < \epsilon/2$, by compacity of the torus, there exists a finite collection of balls $\mathfrak{B}$ of radius η covering $\mathbb{T}^d_{[a]}$. Since $\eta < \epsilon/2$, if x belongs to some ball $B \in \mathfrak{B}$ then $B \subset B(x, \epsilon)$, hence

$$\bigcap_{B\in\mathfrak{B}} (\omega(B) \neq 0) \subset \left(\mathcal{U}_\epsilon(\omega) = \mathbb{T}^d_{[a]}\right).$$

Thus,

$$\begin{aligned}\mathbf{P}_\lambda\left(\mathcal{U}_\epsilon(\omega) \neq \mathbb{T}^d_{[a]}\right) &\leq \mathbf{P}_\lambda\left(\bigcup_{B\in\mathfrak{B}} (\omega(B) = 0)\right)\\ &\leq K\exp(-\lambda(2\eta)^d) \xrightarrow{\lambda\to\infty} 0.\end{aligned}$$

Moreover, by the nerve lemma

$$\left(\mathcal{U}_\epsilon(\omega) = \mathbb{T}^d_{[a]}\right) \subset \bigcap_{i=0}^{d} \left(\beta_i(\mathcal{C}_\epsilon) = \beta_i(\mathbb{T}^d_{[a]})\right),$$

and the result follows. □

The following lemma is straightforward.

Lemma 14. *Let k_1, k_2 and k_3 be real positive constants and $f : \mathbf{R}^2_+ \to \mathbf{R}$ defined as*

$$f(x,y) = \exp\left(-\frac{k_1 - x}{2k_2}\log\left(1 + \frac{(k_1 - x)k_2}{k_3 y}\right)\right).$$

Then, for $k_1 - x > 0$, the function is strictly increasing with respect to x and with respect to y.

Theorem 15. *For $y > \lambda a^d$, we have*

$$\mathbf{P}_\lambda(\beta_0 \geq y) \leq \exp\left(-\frac{y - \lambda a^d}{2}\log\left(1 + \frac{y - \lambda a^d}{(2^d - 1)^2\lambda}\right)\right).$$

Proof. To apply Theorem 8, we need to evaluate $\max(D\beta_0)$ and $\|D\beta_0\|_{L^\infty(\Omega,L^2(Y))}$. Since there are more points than connected components, $\mathbf{E}_\lambda[\beta_0] \leq \mathbf{E}_\lambda[N_0] = \lambda a^d$. According to the definition of D, $\max(D\beta_0)$ is the maximum variation of β_0 induced by the addition of an arbitrary point. If this point is at a distance less than ϵ from ω, then $D\beta_0 \leq 0$, otherwise, $D\beta_0 = 1$, so $\max(D\beta_0) = 1$. Besides, this added point can join at most two connected components in each dimension, so in d dimensions it can join at most 2^d connected component, which means that $D\beta_0$ ranges from $-(2^d - 1)$ to 1, and then

$$\|D\beta_0\|_{L^\infty(\Omega,L^2(Y))} \leq \lambda \max|D\beta_0|^2 = \lambda(2^d - 1)^2.$$

Using Lemma 14 and Theorem 8, we get:

$$\mathbf{P}_\lambda(\beta_0 \geq y) \leq \exp\left(-\frac{y - \lambda a^d}{2}\log\left(1 + \frac{y - \lambda a^d}{(2^d - 1)^2\lambda}\right)\right),$$

for $y > \lambda a^d \geq \mathbf{E}_\lambda[\beta_0]$. □

5. Convergence

Let Γ be an arbitrary connected simplicial complex containing n points and X be a random simplicial complex in a compact set B generated by the Poisson point process ω. The number of occurrences of Γ in X is denoted as G_Γ. In what follows, we denote by $\mathbf{V}_\lambda[X]$ the variance of the random variable X under the probability distribution $\mathbf{P}_\lambda$. In this section, c is a constant which may vary from line to line, may depend on everything but λ.

Lemma 16. *The random variable G_Γ has a chaos representation given by:*

$$G_\Gamma = \sum_{i=0}^{n} I_i(f_i),$$

where for any $i \in \{1, \cdots, n\}$,

$$f_i(x_{i+1}, \cdots, x_n) = \binom{n}{i} \lambda^{n-i} \int_{B^{n-i}} f(x_1, \cdots, x_n)\ dx_1 \dots\ dx_{n-i}, \tag{20}$$

is a bounded symmetric function.

Proof. Let k be the number of 1-simplices of G_Γ, $1 \leq m \leq k$ and $1 \leq l \leq k$. Besides, let $1 \leq i_m \leq n$ and $1 \leq j_l \leq n$ such that $i_m \neq j_m$, $\{i_m, j_m\} \neq \{i_l, j_l\}$ if $m \neq l$. Finally,

$$(i_{m+1} \in \{i_1, \cdots, i_m\} \cup \{j_1, \cdots, j_m\}) \cup (j_{m+1} \in \{i_1, \cdots, i_m\} \cup \{j_1, \cdots, j_m\}).$$

We define J_m recursively as:

$$J_1 = \{(i_1, j_1)\},\ J_m = J_{m-1} \cup \{(i_m, j_m)\},$$

Let

$$h(x_1, \cdots, x_n) = \frac{1}{c_\Gamma} \prod_{(i,j) \in J_k} \mathbf{1}_{[\|x_i - x_j\| \leq \epsilon]},$$

where c_Γ is the number of permutations of $\{x_1, \cdots, x_n\}$ such that

$$h(x_1, \cdots, x_n) = h(x_{\sigma(1)}, \cdots, x_{\sigma(n)}),$$

and let $f(x_1, \cdots, x_n)$ be the symmetrization of $h(x_1, \cdots, x_n)$. This means that

$$G_\Gamma = \sum_{\substack{x_1, \dots, x_n \in \omega \\ x_i \neq x_j i f i \neq j}} f(x_1, ..., x_n) = \int_{\Delta_n} f(x_1, ..., x_n)\ \mathrm{d}\omega(x_1) \cdots\ \mathrm{d}\omega(x_n).$$

Using the binomial expansion and some algebra, we obtain:

$$G_\Gamma = \sum_{i=0}^{n} \int_{\Delta_i} \left(\binom{n}{i} \int_{\Delta_{n-i}} f(x_1, \cdots, x_n) \lambda\ \mathrm{d}x_1 \cdots \lambda dx_{n-i} \right) \\ (\ \mathrm{d}\omega(x_{n-i+1}) - \lambda\ \mathrm{d}x_{n-i+1}) \cdots (\ \mathrm{d}\omega(x_n) - \lambda\ \mathrm{d}x_n),$$

which is equal to $\sum_{i=0}^{n} I_i(f_i)$ taking f_i as in (20). To conclude the proof, it suffices to note that, since B is a compact set and h is bounded, f_i is bounded. □

Lemma 17. *For any Γ connected simplicial complex containing n points, for λ large enough,*

$$\mathbf{E}_\lambda [G_\Gamma] \leq c\lambda^n \text{ and } \mathbf{V}_\lambda [G_\Gamma] \leq P_\Gamma^{2n-1}(\lambda).$$

where $P_\Gamma^{2n-1}(\lambda)$ is a polynomial on λ of degree $2n-1$ depending on Γ.

Proof. Using (16) and the chaos properties, we obtain

$$\mathbf{E}_\lambda [G_\Gamma] = \lambda^n \int_{\Delta_n} f(x_1, \cdots, x_n)\ \mathrm{d}x_1 \cdots\ \mathrm{d}x_n \leq c\lambda^n,$$

since f is bounded. Furthermore,

$$\begin{aligned}\mathbf{V}_\lambda[G_\Gamma] &= \sum_{i=1}^{n} i!\|f_i\|^2_{L^2(B,\lambda)} \\ &= \sum_{i=1}^{n} i! \int_{\Delta_i} \left(\lambda^{n-i}\binom{n}{i}\int_{B^{n-i}} f(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_{n-i}\right)^2 \lambda\,\mathrm{d}x_1\ldots\lambda\,\mathrm{d}x_i \\ &= \sum_{i=1}^{n} i!\,\lambda^{2n-i}\int_{\Delta_i}\left(\binom{n}{i}\int_{B^{n-i}} f(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\,\mathrm{d}x_{n-i}\right)^2 \mathrm{d}x_1\ldots\,\mathrm{d}x_i.\end{aligned}$$

and since f is bounded, $\mathbf{V}_\lambda[G_\Gamma]$ is a polynomial of degree $2n-1$. □

Lemma 18. *For λ large enough, if $k>1$,*

$$\mathbf{E}_\lambda\left[(I_k(1))^2\right] < c\lambda^{2k-1},$$

and $\mathbf{E}_\lambda\left[(I_k(1))^2\right]$ is constant if $k=0$.

Proof. The proof is trivial for the case $k=0$. If $k\geq 1$, for $i\leq k$ we have

$$\int_{\Delta_i}\mathrm{d}\omega(x_1)\cdots\,\mathrm{d}\omega(x_i) = \prod_{j=0}^{i-1}(\omega(B)-j),$$

so we can rewrite $I_k(1)$ as follows:

$$I_k(1) = \sum_{i=0}^{k}\left[\binom{k}{i}(-\lambda S(B))^i\prod_{j=0}^{k-i-1}(\omega(B)-j)\right].$$

Thus, $\mathbf{E}_\lambda\left[(I_k(1))^2\right]$ can be written as

$$\mathbf{E}_\lambda\left[(I_k(1))^2\right] = \mathbf{E}_\lambda\left[(\omega(B)-\lambda S(B))^{2k} + \sum_{2\leq i+j\leq 2k-1} c_{i,j}\omega(B)^i(\lambda S(B))^j\right],$$

where the $c_{i,j}$ are integer constants.

If we differentiate the k-th central moment

$$\mathbf{E}[(N-\lambda')^k] = \sum_{i=0}^{\infty}(r-\lambda')^k e^{-\lambda'}\frac{(\lambda')^i}{i!}$$

of a random variable N distributed as Poisson with mean λ', with respect to λ' we find the following recurrence:

$$\mathbf{E}[(N-\lambda')^{k+1}] = \lambda'\left(\frac{d\mathbf{E}[(N-\lambda')^k]}{d\lambda'} + k\mathbf{E}[(N-\lambda')^{k-1}]\right).$$

Hence, using induction we can show that $\mathbf{E}[(N-\lambda')^k]$ is a polynomial on λ with maximum degree $\lfloor k/2\rfloor$, for $k>1$. Since $\mathbf{E}_\lambda\left[\omega(B)^i\right]$ is the Bell Polynomial of degree i on λ, it follows straightforwardly that the polynomial

$$\mathbf{E}_\lambda\left[\sum_{2\leq i+j\leq 2k-1} c_{i,j}\omega(B)^i(\lambda S(B))^j\right]$$

has degree at most $2k-1$, and the proof is thus complete. □

Definition 6. *Let f_i, g_j and h_k be, respectively, functions of i-th, j-th and k-th chaos of the Wiener-Poisson decomposition of some square integrable function of ω. For $0 \leq s \leq 2(n \wedge m)$, we define*

$$f_i \star_s g_j = \sum_{s \leq 2n \leq 2(s \wedge i \wedge j \wedge)} n! \binom{i}{n} \binom{j}{n} f_i \circ_n^{s-n} g_j.$$

For $0 \leq r \leq 2((i+j-s) \wedge k)$, we abuse of the notation to write

$$h_k \star_r (f_n \star_s g_m) = h_k \star_r f_n \star_s g_m.$$

Lemma 19. *If $|f_n(x_1, \cdots, x_n)|$ is bounded by a positive real c, then*

$$\mathbf{E}_\lambda \left[I_n(f_n)^2 \right] \leq c^2 \mathbf{E}_\lambda \left[I_n(1)^2 \right].$$

Proof. We use the isometry formula given by Eq. ((6)), so

$$\begin{aligned} \mathbf{E}_\lambda \left[I_n(f_n)^2 \right] &= n! \|f_n\|_{L^2(\lambda)^{\circ n}} \\ &= n! \int_{B^n} f_n^2(x_1, \cdots, x_n) \lambda \, \mathrm{d}x_1 \cdots \lambda \, \mathrm{d}x_n \\ &\leq n! \int_{B^n} c^2 \lambda \, \mathrm{d}x_1 \cdots \lambda \, \mathrm{d}x_n \\ &= c^2 \mathbf{E}_\lambda \left[I_n(1)^2 \right], \end{aligned}$$

and the proof is complete. □

Theorem 20. *Let $F = \frac{G_\Gamma - \mathbf{E}_\lambda[G_\Gamma]}{\sqrt{\mathbf{V}_\lambda[G_\Gamma]}}$, then, for λ large enough,*

$$\int_B \mathbf{E}_\lambda \left[|D_t F|^2 |D_t L^{-1} F| \right] \lambda \, dt \leq \frac{c}{\lambda^{1/2}}.$$

Proof. Provided that G_Γ has n points, Lemma 16 shows that $G_\Gamma = \sum_{i=0}^n I_n(f_n)$, so

$$D_t F = \frac{1}{\sqrt{\mathbf{V}_\lambda [G_\Gamma]}} \sum_{i=1}^n i I_{i-1}(f_i(*,t)),$$

$$D_t L^{-1} F = \frac{1}{\sqrt{\mathbf{V}_\lambda [G_\Gamma]}} \sum_{i=1}^n I_{i-1}(f_i(*,t)).$$

Let us define

$$g_{i-1} = \frac{f_i(*,t)}{\lambda^{n-i}}.$$

According to Eq. (20), we note that g_i does not depend on λ. Using the triangular inequality, we have

$$|D_t F|^2 |D_t L^{-1} F| \leq \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \frac{\lambda^{3n-3-i-j-k}(i+1)(j+1)}{\mathbf{V}_\lambda [G_\Gamma]^{\frac{3}{2}}} |I_i(g_i) I_j(g_j) I_k(g_k)|.$$

Then, we apply twice the chaos expansion and use again the triangular inequality to obtain:

$$|D_t F|^2 |D_t L^{-1} F| \leq \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \sum_{s=0}^{2(i \wedge j)} \sum_{r=0}^{2((i+j-s) \wedge k)} \frac{\lambda^{3n-3-i-j-k}(i+1)(j+1)}{\mathbf{V}_\lambda [G_\Gamma]^{\frac{3}{2}}} \times |I_{i+j+k-s-r}(g_i \star_r g_j \star_s g_k)|,$$

Since f_i is bounded, g_i is bounded as so $g_i \star_r g_j \star_s g_k$ for i, j, k, r, s in the range of their indexes above. We define

$$c(i,j,k,r,s) = \sup\{g_i \star_r g_j \star_s g_k\}(i+1)(j+1),$$

and we use Jensen's inequality and Lemma 19 to write

$$\begin{aligned}
&\mathbf{E}_\lambda\left[|D_tF|^2|D_tL^{-1}F|\right]\\
&\leq \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\sum_{s=0}^{2(i\wedge j)}\sum_{r=0}^{2((i+j-s)\wedge k)} \frac{\lambda^{3n-3-i-j-k}}{\mathbf{V}_\lambda\left[G_\Gamma\right]^{\frac{3}{2}}}\mathbf{E}_\lambda\left[\left(I_{i+j+k-s-r}(g_i\star_r g_j\star_s g_k)\right)^2\right]^{\frac{1}{2}}\\
&\leq \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\sum_{s=0}^{2(i\wedge j)}\sum_{r=0}^{2((i+j-s)\wedge k)} \frac{\lambda^{3n-3-i-j-k}c(i,j,k,r,s)}{\mathbf{V}_\lambda\left[G_\Gamma\right]^{\frac{3}{2}}}\mathbf{E}_\lambda\left[\left(I_{i+j+k-s-r}(1)\right)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Using Lemmas 17 and 18 we obtain:

$$\begin{aligned}
\int_B \mathbf{E}_\lambda\left[|D_tF|^2|D_tL^{-1}F|\right]\lambda\,\mathrm{d}t &\leq \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\sum_{s=0}^{2(i\wedge j)}\sum_{r=0}^{2((i+j-s)\wedge k)} c(i,j,k,r,s)\times\\
&\quad \frac{\lambda^{3n-3-i-j-k}}{\mathbf{V}_\lambda\left[G_\Gamma\right]^{\frac{3}{2}}}\mathbf{E}_\lambda\left[\left(I_{i+j+k-s-r}(1)\right)^2\right]^{1/2}\int_B\lambda\,\mathrm{d}t\\
&\leq \frac{c\lambda^{3n-3}}{\mathbf{V}_\lambda\left[G_\Gamma\right]^{\frac{3}{2}}}\int_B\lambda\,\mathrm{d}t\leq\frac{c}{\lambda^{1/2}},
\end{aligned}$$

concluding the proof. □

Theorem 21. *Let*

$$F=\frac{G_\Gamma-\mathbf{E}_\lambda\left[G_\Gamma\right]}{\sqrt{Var(G_\Gamma)}}.$$

Then, when λ is large enough

$$\mathbf{E}_\lambda\left[\left|1-\langle DF,DL^{-1}F\rangle_{L^2(\lambda)}\right|\right]\leq\frac{c}{\lambda^{1/2}},$$

for some constant c.

Proof. The expressions of D_tF and $D_tL^{-1}F$ are given by

$$D_tF=\frac{1}{\sqrt{\mathbf{V}_\lambda\left[G_\Gamma\right]}}\sum_{i=1}^n iI_{i-1}(f_i(*,t)),$$

and

$$D_tL^{-1}Z=\frac{1}{\sqrt{\mathbf{V}_\lambda\left[G_\Gamma\right]}}\sum_{i=1}^n I_{i-1}(f_i(*,t)).$$

The inner product $\langle D_tL^{-1}F,D_tF\rangle_{L^2(\lambda)}$ is expressed by:

$$\langle D_tL^{-1}F,D_tF\rangle_{L^2(\lambda)}=\frac{1}{\mathbf{V}_\lambda\left[G_\Gamma\right]}\int_B\sum_{j=1}^n\sum_{i=1}^n iI_{i-1}(f_i(*,t))I_{j-1}(f_j(*,t))\lambda\,\mathrm{d}t.$$

Then,

$$\begin{aligned}
\langle D_tL^{-1}Z,D_tZ\rangle_{L^2(\lambda)} &= \frac{1}{\mathbf{V}_\lambda\left[G_\Gamma\right]}\sum_{j=1}^n\sum_{i=1}^n i\int_B I_{i-1}(f_i(*,t))I_{j-1}(f_j(*,t))\lambda\,\mathrm{d}t\\
&= \frac{1}{\mathbf{V}_\lambda\left[G_\Gamma\right]}\int_B I_0(f_1(t)^2\lambda\,\mathrm{d}t\\
&+ \frac{1}{\mathbf{V}_\lambda\left[G_\Gamma\right]}\sum_{\substack{1\leq i\leq n\\1\leq j\leq n\\(i,j)\neq(1,1)}} i\int_B I_{i-1}(f_i(*,t))I_{j-1}(f_j(*,t))\lambda\,\mathrm{d}t.
\end{aligned}$$

Defining g_{i-1} as in Theorem 20 and using the chaos expansion, we get:

$$\begin{aligned}\langle D_t L^{-1}F, D_t F\rangle_{L^2(\lambda)} &= \frac{\|f_1\|^2_{L^2(\lambda)}}{\mathbf{V}_\lambda[G_\Gamma]} + \frac{1}{\mathbf{V}_\lambda[G_\Gamma]}\sum_{i=2}^{n} i(i-1)!\int_B \|f_n(*,t)\|^2_{L^2(\lambda)}\,\lambda\,\mathrm{d}t \\ &+ \sum_{i\geq 1}^{n-1}(i+1)\frac{\lambda^{2n-2i-2}}{\mathbf{V}_\lambda[G_\Gamma]}\int_B \sum_{s=0}^{2(i-1)} I_{2i-s}(g_i\star_s g_i)\lambda\,\mathrm{d}t \\ &+ \sum_{\substack{0\leq i\leq n-1\\ 0\leq j\leq n-1}}(i+1)\frac{\lambda^{2n-i-j-2}}{\mathbf{V}_\lambda[G_\Gamma]}\int_B \sum_{s=0}^{2(i\wedge j)} I_{i+j-s}(g_i\star_s g_j)\lambda\,\mathrm{d}t.\end{aligned}$$

Since

$$\begin{aligned}\int_B \|f_i(*,t)\|^2_{L^2(\lambda)}\,\lambda\,\mathrm{d}t &= \int_B\left(\int_{B^{i-1}} f_i^2(t_1,...,t_{i-1},t)\lambda\,\mathrm{d}t_1...\lambda\,\mathrm{d}t_{i-1}\right)\lambda\,\mathrm{d}t \\ &= \int_{B^i} f_i^2(t_1,...,t_{i-1},t)\lambda\,\mathrm{d}t_1...\lambda\,\mathrm{d}t_{i-1}\lambda\,\mathrm{d}t = \|f_i\|_{L^2(\lambda)},\end{aligned}$$

and given the isometry formula

$$\begin{aligned}\mathbf{V}_\lambda[G_\Gamma] &= \|G_\Gamma\|_{L^2\Omega} - \mathbf{E}_\lambda[G_\Gamma]^2 \\ &= \sum_{i=0} n!\|f_i\|^2_{L^2(B^i)} - \|f_0\|^2 = \sum_{i=1} n!\|f_i\|^2_{L^2(\lambda)},\end{aligned}$$

we have

$$\frac{\|f_1\|^2_{L^2(\lambda)}}{\mathbf{V}_\lambda[G_\Gamma]} + \frac{1}{\mathbf{V}_\lambda[G_\Gamma]}\sum_{i=2}^{n} i(i-1)!\int_B \|f_i(*,t)\|^2_{L^2(\lambda)}\,\lambda\,\mathrm{d}t = 1.$$

Hence

$$\langle D_t L^{-1}Z, D_t Z\rangle_{L^2(\lambda)} = 1 + \sum_{\substack{0\leq i\leq n-1\\ 0\leq j\leq n-1\\ (i,j)\neq 1}}\sum_{\substack{0\leq s\leq 2(i\wedge j)\\ s\neq 2i \text{ if } i=j}}(i+1)\frac{\lambda^{2n-i-j-2}}{\mathbf{V}_\lambda[G_\Gamma]}\int_B I_{i+j-s}(g_i\star_s g_j)\lambda\,\mathrm{d}t.$$

Let $c(i,j,s)$ be defined as

$$c(i,j,s) = \sup\{g_i\star_s g_j\}(i+1).$$

Then, we use the triangular inequality, Jensen's inequality and Lemma 19 to obtain:

$$\begin{aligned}&\mathbf{E}_\lambda\left[|1-\langle DF, DL^{-1}F\rangle_{L^2(\lambda)}|\right] \\ &\leq \mathbf{E}_\lambda\left[\sum_{\substack{0\leq i\leq n-1\\ 0\leq j\leq n-1\\ (i,j)\neq 1}}\sum_{\substack{0\leq s\leq 2(i\wedge j)\\ s\neq 2i \text{ if } i=j}}(i+1)\frac{\lambda^{2n-i-j-2}}{\mathbf{V}_\lambda[G_\Gamma]}\int_B |I_{i+j-s}(g_i\star_s g_j)|\,\lambda\,\mathrm{d}t\right] \\ &\leq \sum_{\substack{0\leq i\leq n-1\\ 0\leq j\leq n-1\\ (i,j)\neq 1}}\sum_{\substack{0\leq s\leq 2(i\wedge j)\\ s\neq 2i \text{ if } i=j}}\frac{(i+1)\lambda^{2n-i-j-2}}{\mathbf{V}_\lambda[G_\Gamma]}\int_B \mathbf{E}_\lambda\left[(I_{i+j-s}(g_i\star_s g_j))^2\right]^{\frac{1}{2}}\lambda\,\mathrm{d}t \\ &\leq \sum_{\substack{0\leq i\leq n-1\\ 0\leq j\leq n-1\\ (i,j)\neq 1}}\sum_{\substack{0\leq s\leq 2(i\wedge j)\\ s\neq 2i \text{ if } i=j}} c(i,j,s)\frac{\lambda^{2n-i-j-2}}{\mathbf{V}_\lambda[G_\Gamma]}\mathbf{E}_\lambda\left[(I_{i+j-s}(1))^2\right]^{1/2}\int_B \lambda\,\mathrm{d}t.\end{aligned}$$

Finally, using Lemmas 18 and 17, there is a constant c such that:

$$\mathbf{E}_\lambda\left[|1-\langle DF, DL^{-1}F\rangle_{L^2(\lambda)}|\right] \leq \frac{c}{\lambda^{1/2}}$$

for λ large enough. □

Theorem 22. *There exists a constant c such that, for λ large enough, the Wasserstein distance between $F = \frac{G_\Gamma - \mathbf{E}_\lambda[G_\Gamma]}{\sqrt{Var(G_\Gamma)}}$ and $\mathcal{N}(0,1)$ is given by:*

$$d_W(F, \mathcal{N}(0,1)) \leq \frac{c}{\lambda^{1/2}}.$$

Proof. The proof comes straightforwardly from Theorem 3.1 as stated in [8]:

$$d_W(F, \mathcal{N}(0,1)) \leq \mathbf{E}_\lambda \left[|1 - \langle DF, DL^{-1}F \rangle_{L^2(\lambda)}| \right] + \int_B \mathbf{E}_\lambda \left[|D_t F|^2 |D_t L^{-1} F| \right] \lambda \, \mathrm{d}t,$$

which we can use since f_n is bounded, so $F \in \operatorname{Dom} D$. We use theorems 20 and 21 in the first and second terms, respectively.

□

6. Discussion and Conclusion

Through this work, we successfully obtained the means of the number of high order interactions, named simplices and the Euler characteristic. Except for the maximum norm, the other assumptions are quite reasonable and are justified for real systems. The randomness of the positions reflect the lack of capacity of choosing the position where they are deployed, while the randomness of the number of sensors comprise various random factors that can happens with a sensor that suppose to be in the target region (for instance, the sensor can fall outside the region or can shut off due the lack of energy). The fact that they fall over a torus, can be justified if we consider that we can warrant a total coverage in the box $[0,a]^d$ if the Euler characteristic is zero in the Torus, which would not be possible to find if we consider that points fall directly over $[0,a]^d$. Besides, the behavior in the torus or in the box is close from each other when we consider small cover of each point. Maybe we can criticize the use of the maximum norm, which is not physically realistic, but this norm allows us to find closed form formulas results and can represent bounds for the Euclidean one. If $\mathbf{E}_\lambda[N'_k]$ and $\mathbf{E}_\lambda[\chi']$ represent, respectively, the mean number of k-simplices and the mean of the Euler characteristic, using the Euclidean norm we obtain:

$$\frac{\lambda^{k+1} a^d}{(k+1)!} \frac{(k+1)^d \left((2\epsilon)^k\right)^d}{\sqrt{d}^{d(k)}} \leq \mathbf{E}_\lambda[N'_k] \leq \frac{\lambda^{k+1} a^d}{(k+1)!} (k+1)^d \left((2\epsilon)^k\right)^d$$

$$\left(\frac{a\sqrt{d}}{2\epsilon}\right)^d e^{-\lambda(2\epsilon)^d \left(-B_d\left(-\lambda \frac{(2\epsilon)^d}{\sqrt{d}^d}\right)\right)} \leq \mathbf{E}_\lambda[\chi'] \leq \left(\frac{a}{2\epsilon}\right)^d e^{-\lambda(2\epsilon)^d (-B_d(-\lambda(2\epsilon)^d))}$$

From the properties obtained of the Euler characteristic mean and based in some simulations, we conjecture that, in a random simplicial complex as defined in this work, there is always two main kinds of holes in this complex, β_i and β_{i+1}. So, for instance, consider figure 6 where points are placed in 5 dimensions. When λ is small, in average, the components are isolated from the others, so $\beta_0 > 0$ and we do not have other kinds of holes, so β_0 is the dominating Betti number. If λ is increased (which means, in average, increase the number of points), the components connect with each other, decreasing β_0 and we begin to form the first cycles, so β_0 and β_1 are the dominating Betti numbers. Increasing λ even more, the complex becomes completely connected and we have a large number of cycles. For λ even larger, those cycles begin to vanish and we have the first voids. Then, we follow this reasoning until all the region is completely covered and $\chi = 0$.

As future works we propose to find the expressions of the second moments of his quantities. This introduces a large degree of difficulty but it is probably possible to

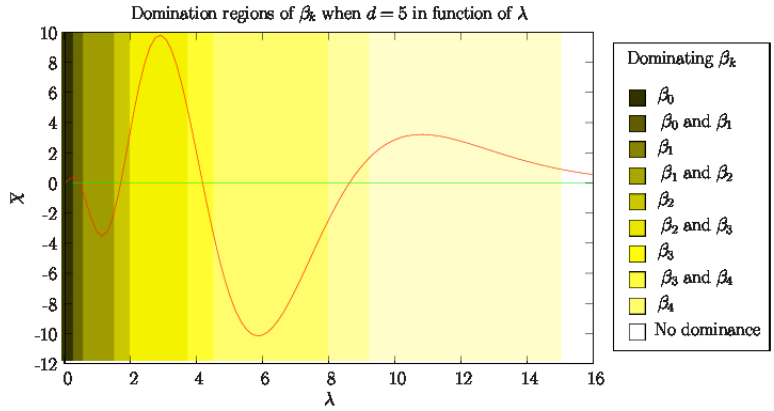


FIGURE 6. Behavior of χ with respect to λ.

do that. An interesting result could be to find the expression of the nth moment of such quantities, so we could obtain their distributions. If it is not possible, we could search the behavior of s_k and χ in the extreme cases, especially when $\lambda \to \infty$. Another interesting result could be achieved if we take into account the topology of the network in function of time if sensors have a random lifetime.

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