Locally stationary Hawkes processes

François Roueff, Rainer Von Sachs, Laure Sansonnet

Time-frequency analysis of locally stationary Hawkes processes

François Roueff*

Institut Mines-Télécom ; Télécom ParisTech ; CNRS LTCI
46, rue Barrault, 75634 Paris, France

Rainer von Sachs*

Institut de statistique, biostatistique et sciences actuarielles (ISBA) IMMAQ
Université catholique de Louvain, Voie du Roman Pays 20, boite L1.04.01, B-1348 Louvain-la-Neuve, Belgium

Laure Sansonnet*

Équipe Stat & Génome, AgroParisTech
16 rue Claude Bernard F-75231, Paris Cedex 05, France

Abstract

This paper addresses the generalisation of stationary Hawkes processes in order to allow for a time-evolving second-order analysis. Motivated by the concept of locally stationary autoregressive processes, we apply however inherently different techniques to describe the time-varying dynamics of self-exciting point processes. In particular we derive a stationary approximation of the Laplace transform of a locally stationary Hawkes process. This allows us to define a local intensity function and a local Bartlett spectrum which can be used to compute approximations of first and second order moments of the process. We complete the paper by some insightful simulation studies.

Keywords: Locally stationary processes, Hawkes processes, Bartlett spectrum, time frequency analysis, point processes
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1. Introduction

Introductory work on Hawkes processes, an important class of self-exciting point processes, and in particular on the analysis of its spectrum, the Bartlett spectrum (i.e. the Fourier transform of the autocovariance of the process) is to be found mainly in the following seminal references: [1, 2, 3, 4]. A. Hawkes ([1]) was the first to provide for the definition of a point process with a self-exciting behaviour. Intuitively similar to a Poisson process, the conditional intensity function of a Hawkes process is however stochastic as it depends on its own past events. Whereas Hawkes' model was introduced to reproduce the ripple effects generated after the occurrence of an earthquake, applications of this model have become since then really

*Corresponding author

Email addresses: roueff@telecom-paristech.fr (François Roueff), rvs@uclouvain.be (Rainer von Sachs), laure.sansonnet@agroparistech.fr (Laure Sansonnet)

numerous in many and diverse fields such as seismology (see e.g., [5], for a recent review), biology ([6] on genome analysis) or neuroscience ([7] on brain data analysis), to name but a few. Recently, this model is also being widely used in finance where self-exciting processes led to many applications such as microstructure dynamics ([8]), order arrival rate modelling and high-frequency data ([9], [10], [11]), financial price modelling across scales ([12]), and many others. For a really comprehensive list of applications of Hawkes processes (including very recent applications on limit order book modelling as in [13]) we refer also to the recent PhD thesis of A. Iuga ([14]).

In this paper, we contribute by generalising existing models of stationary Hawkes processes (i.e. with time-invariant second order structure) to model and capture their time-varying dynamics. To begin with, we recall some basic features of a stationary linear Hawkes process with fertility p defined on the positive half-line. The conditional intensity function $\lambda(t)$ of such a process is driven by the fertility function taken at the time distances to previous points of the process, i.e. $\lambda(t)$ is given by

$$\lambda(t) = \nu + \int_{-\infty}^{t^-} p(t-s)\; N(\mathrm{d}s) \;=\; \nu + \sum_{t_i<t} p(t-t_i)\;. \tag{1}$$

Here the first display is to be read as the stochastic Stieltjes integral of the "fertility" function p with respect to the counting process $N(t)$. As will be derived in more detail in Section 2.2 below, linear self-exciting processes can also be viewed as clusters of point processes. For a classical Hawkes process on the real line, these point processes are Poisson processes, and each event is one of two types: an immigrant process or an offspring process. The immigrants follow a Poisson process and define the centers of so-called Poisson clusters. As immigrants and offsprings can be referred to as "main shocks" and "after shocks" respectively, an interesting interpretation arises which is useful not only in seismology but also in high-frequency finance. We refer to [10] who exploit that Hawkes processes capture the dynamics in financial point processes remarkably well, and hence, their cluster property can serve as a reasonable description of the timing structure of events on financial markets.

In the more general case of spatial Hawkes processes with values in $\mathbb{R}^\ell$, the cluster dynamics and interpretation remains the same, except that the immigrants now constitute a more general spatial point process whereas conditionally on a given point s of the Hawkes process ("population"), independently of the previous history, the resulting offspring process is Poisson as it is in the one-dimensional case, again with non-negative (and locally integrable) fertility rate $p(\cdot - s)$. Spatial Hawkes processes provide natural models for e.g. a population of reproducing individuals or the development of an epidemic.

For spatial Hawkes process in general we refer, e.g., to [15] whereas [4] consider (unmarked) stationary spatial Hawkes processes and obtain the Bartlett spectrum, assuming the existence of the Bartlett spectrum of the immigrant process.

In this paper we develop a new non-parametric model of a generalised (temporal, spatial, or spatio-temporal) Hawkes process with a view on analysis of its Bartlett spectrum. Indeed, the challenge and motivation for our new approach comes from the fact that these days, in many of the afore-mentioned applications such as genomics or high-frequency data analysis practitioners have to face (potentially very) long data stretches. Hence the assumption of a stationary model is no more realistic and needs to be given up. In terms of spectral analysis of Hawkes processes, this means that a time-frequency analysis is required which calls for the development of a mathematical model: this model should allow for a rigorous definition of a generalised, i.e. time-varying, Bartlett spectrum. In this paper we adopt the point of view of local stationarity as introduced by Dahlhaus (see, e.g., [16]) in order to accomplish this task. Here the idea is that the observed Hawkes process is embedded into a doubly-indexed sequence of processes which, as sample size T becomes larger and larger, can locally be better and better approximated by a stationary Hawkes model. Similarities to the treatment of locally stationary autoregressive processes exist, e.g. already formally by letting the fertility function $p(t-s)$ in equation (1) now depend explicitly on time taking the form $p(t-s;t)$, akin the time-dependency of the autoregressive coefficients of a locally stationary process (see our formal development in Section 2.4). However as the dynamics of self-exciting point processes are different from autoregression on the real line, the techniques employed here are inherently different. In particular we derive a stationary approximation of the Laplace transform of the underlying non-stationary Hawkes process by a local Laplace transform. This allows us to define a local intensity function and and a local Bartlett spectrum of the locally stationary Hawkes process. We show how those are used to compute in particular approximations of first and second order moments of the process, including rates of convergence. However, our derivations more generally allow for treatment of all its moments (under suitable and, since the Laplace function characterizes the distribution of a point process uniquely, we can also derive convergence in distribution of the non-stationary towards the locally approximating stationary Hawkes process. We complete the paper by providing for some numerical studies where we simulate some insightful examples of Hawkes processes with time-varying intensity and Bartlett spectrum, respectively. We also indicate empirically how to estimate these quantities from sampled data. The development of some asymptotic estimation theory using the new framework presented in this paper is left for future work.

This paper is organised as follows. Section 2 introduces some notation used throughout the paper and the formal definitions of non-stationary and locally stationary Hawkes processes, as well as the assumptions related to these definitions. The main results are to be found in Section 3, namely a local approximation of the Laplace functional of a locally stationary Hawkes process by that of a stationary one. We also explain how to derive approximations of cumulants and of the mean density. In Section 4, we focus on the one-dimensional case and develop the notion of a local Bartlett spectrum, also discussing how to estimate this quantity from data. This corresponds to a time frequency analysis for non-stationary point processes. Section 5 provides some numerical experiments illustrating our approach. Finally, Section 6 contains the proofs of the main results. A postponed proof and a useful lemma have been placed in Appendix Appendix A for convenience.

2. Main definitions and assumptions

2.1. Conventions and notation

Throughout the paper, ℓ is a positive integer and we work with point processes and measures on the space $\mathbb{R}^\ell$ endowed with the Borel σ-field. For any $x \in \mathbb{R}^\ell$, we denote by $|x|$ the Euclidean norm of x.

A point process is identified with a random measure with discrete support, $N = \sum_k \delta_{T_k}$ typically, where δ_t is the Dirac measure at point t and $\{T_k\}$ the corresponding (countable) random set of points. We use the notation $\mu(g)$ for a measure μ and a function g to express $\int g \, \mathrm{d}\mu$ when convenient. In particular, for a measurable set A, $\mu(A) = \mu(1_A)$ and for a point process N, $N(g) = \sum_k g(T_k)$. The shift operator of lag t is denoted by S^t. For a set A,

$S^t(A) = \{x - t,\, x \in A\}$ and for a function g, $S^t(g) = g(\cdot + t)$, so that $S^t(1_A) = 1_{S^t(A)}$. One can then compose a measure μ with S^t, yielding for a function g, $\mu \circ S^t(g) = \mu(g(\cdot + t))$.

We also need some notation for the functional norms which we deal with in this work. Usual L^q-norms are denoted by $|h|_q$,

$$|h|_q = \left(\int |h|^q \right)^{1/q} ,$$

for $q \in [1, \infty)$ and $|h|_\infty$ is the essential supremum on $\mathbb{R}^\ell$,

$$|h|_\infty = \operatorname*{ess\,sup}_{t \in \mathbb{R}^\ell} |h(t)| \ .$$

We also use the following weighted L^1 norm to control the decay of a function $h : \mathbb{R}^\ell \to \mathbb{R}$ and a given positive exponent β,

$$|h|_{(\beta)} := \left| h \times |\cdot|^\beta \right|_1 = \int |h(s)|\, |s|^\beta \, \mathrm{d}s \ .$$

Let now m be a positive integer and U be an open subset of $\mathbb{C}^m$. Define $\mathcal{O}(U)$ be the set of holomorphic functions from U to $\mathbb{R}$. We will use the compact open topology presented in [17, Section 1.4]. The convergence under this topology is equivalent to uniform convergence over all compact subsets of U, and, more importantly, $\mathcal{O}(U)$ endowed with this topology is complete. We denote, for all $h \in \mathcal{O}(U)$ and compact sets $K \subset U$,

$$|h|_{\mathcal{O},K} = \sup_{z \in K} |h(z)| \ .$$

Recall that a holomorphic function h on U is infinitely differentiable on U and that, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, the partial derivative operator $\partial^\alpha = (\partial/\partial z_1)^{\alpha_1} \ldots (\partial/\partial z_m)^{\alpha_m}$ is continuous over $\mathcal{O}(U)$ for the compact open topology. Let us denote, for $r > 0$, the polytorus $T_r^m(z) = \{z' \in \mathbb{C}^m \ : |z'_i - z_i| = r\}$ and the polydisc $P_r^m(z) = \{z' \in \mathbb{C}^m \ : |z'_i - z_i| < r\}$. We have moreover from [17, Theorem 1.3.3] that the partial derivatives satisfies the Cauchy inequality

$$|\partial^\alpha h(z)| \le \frac{\alpha!}{r^\alpha} \sup_{T_r^m(z)} |h(z)| \ , \tag{2}$$

where $\alpha! = \alpha_1! \ldots \alpha_m!$ and $r^\alpha = r^{\alpha_1} \ldots r^{\alpha_m}$. We denote by $\bar{\mathcal{O}}(U)$ the set of $\mathbb{R}^\ell \times U \to \mathbb{R}$ functions h such that, for all $t \in \mathbb{R}^\ell$, $z \mapsto h(t,z)$ belongs to $\mathcal{O}(U)$. The translation operator S^s is extended to this setting by defining $\bar{S}^s$ for any $s \in \mathbb{R}^\ell$ as the operator

$$\bar{S}^s(h) : (t,z) \mapsto h(t+s, z) \ ,$$

that is, we translate h by the lag s only through its first parameter. When $h \in \bar{\mathcal{O}}(U)$, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$, we denote by $\partial_{\mathcal{O}}^\alpha h$ the function obtained by differentiating with respect to the second variable, that is, for all $t \in \mathbb{R}^\ell$ and $z = (z_1, \ldots, z_m) \in U$,

$$\partial_{\mathcal{O}}^\alpha h(t,z) = \left(\frac{\partial}{\partial z_1} \right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial z_m} \right)^{\alpha_m} h(t,z) \ .$$

An immediate consequence of the Cauchy inequality is that for $h \in \bar{\mathcal{O}}(U)$, one can integrate with respect to t and obtain a holomorphic function, providing some simple condition on the integrability of the local supremum, see Lemma 15 for a precise statement.

For any $p \in [1, \infty]$, we further denote by $\bar{\mathcal{O}}_p\,(U)$ the subset of functions $h \in \bar{\mathcal{O}}\,(U)$ such that function $t \mapsto \sup_{z \in K} h(t,z)$ has finite L^p-norm on $\mathbb{R}^\ell$ for all compact set $K \subset U$. We denote

$$|h|_{\bar{\mathcal{O}},K,p} := \left| \sup_{z \in K} |h(\cdot, z)| \right|_p .$$

We also denote by $B_{\bar{\mathcal{O}}}\,(r; K, p)$ the set of all functions $g \in \bar{\mathcal{O}}_p\,(U)$ such that $|g|_{\bar{\mathcal{O}},K,p} < r$. Finally, for a given exponent $\beta > 0$ and a compact set $K \subset U$, we use the following norm for $h \in \bar{\mathcal{O}}\,(U)$,

$$|h|_{\bar{\mathcal{O}},K,(\beta)} = \left| \sup_{z \in K} |h(\cdot, z)| \right|_{(\beta)} .$$

The corresponding balls are denoted, for given $r > 0$,

$$B_{\bar{\mathcal{O}}}\,(r; K, (\beta)) = \left\{ h \in \bar{\mathcal{O}}\,(U) \ : \ |h|_{\bar{\mathcal{O}},K,(\beta)} < r \right\} .$$

2.2. Hawkes processes as cluster processes

Although intuitive, the definition of Hawkes processes through its conditional intensity as in (1) is only adapted to time point processes. A more general approach for defining Hawkes processes applying for points in the space $\mathbb{R}^\ell$ is to see them as a special case of cluster processes. Cluster processes are point processes constructed via conditioning on the realization of a so-called *center process*, usually a Poisson point process, denoted PPP in the sequel (see [3, Section 6.3] for example). We consider here point processes on the space $\mathbb{R}^\ell$.

Let N_c be a PPP with intensity measure μ_c. This is the starting point for the following mechanism as it represents the *immigrants* which appear spontaneously (in fact, later on they will represent those parents which are not generated by the iteration in the *offspring* generation). At each *center point* t of N_c, a point process $N(\cdot|t)$ is generated (we will explain below how these *descendants* of t are generated). The cluster process N is defined as the set of all the immigrants (points of the PPP N_c) and of all the descendants (realizations of the point process $N(\cdot|t)$) generated at each *center point* t of N_c:

$$N(A) = N_c\big(N(A|\cdot)\big), \quad \text{for every bounded } A \text{ in } \mathcal{B}(\mathbb{R}) . \tag{3}$$

Remark 1. Recall our notation : here, we have to do with an integration over *center points* t using the measure N_c. In fact, $N(\cdot|t)$ is called the *component* process (see [3, Definition 6.3.I]) generated at position t and the process N is merely the superposition of all these components when the center points t runs over the support of N_c.

Hawkes processes are cluster processes for which N_c is a PPP and $N(\cdot|t)$ are independent branching processes in which each point t has offspring defined as a PPP with finite intensity measure $\mu(\cdot|t)$. We detail below the iterative scheme for generating all generations of the components $N(\cdot|t)$. For the moment, let us precise that standard Hawkes processes are made stationary by assuming that N_c is a homogeneous PPP on the whole space $\mathbb{R}^\ell$ and $t \mapsto \mu(\cdot|t)$ is shift invariant, $\mu(\cdot|t) = \mu \circ S^t$, where μ is fixed (*i.e.* $\mu = \mu(\cdot|0)$). In this case, the condition $\mu(\mathbb{R}) < 1$ insures that the obtained process has finite intensity (density) $m = \mu_c/(1 - \mu(\mathbb{R}))$ (see [3, Example 6.3(c)]). The second order properties are also derived in this case (see [4, 15] for additional insights). In the following section, we extend the Hawkes model to the non-stationary case by authorizing N_c to be non-homogeneous and $t \mapsto \mu(\cdot|t)$ to be non-shift invariant.

2.3. Non-stationary Hawkes processes

In this section, we consider non-stationary Hawkes processes, namely $t \mapsto \mu_c(t)$ and $t \mapsto \mu(\cdot|t)$ may not be shift invariant. The usual cluster construction still applies in this case. Namely, each component $N(\cdot|t)$ can be constructed as the superposition of point processes defined iteratively. For each center point t,

$$\begin{aligned} N^{(0)}(\cdot|t) &= \delta_t \\ N^{(n+1)}(\cdot|t) &= \int \mathrm{m}^{(n)}(\cdot|s)\, N^{(n)}(\mathrm{d}s|t), \quad \text{for all } n \geq 0 \;, \end{aligned}$$

where $\{\mathrm{m}^{(n)}(\cdot|s),\ s \in \mathbb{R}^\ell,\ n \geq 0\}$ are independent PPPs with respective intensity measure $\mu(\cdot|s)$. The resulting component at center point t is defined as

$$N(\cdot|t) = \sum_{n\geq 0} N^{(n)}(\cdot|t) \;. \tag{4}$$

We observe that for any non-negative (test) function g defined on $\mathbb{R}$, we have

$$\mathbb{E}[N^{(0)}(g|t)] = g(t)$$

and, for all $n \geq 0$,

$$\mathbb{E}\left[N^{(n+1)}(g|t)\right] = \mathbb{E}\left[\mathbb{E}[N^{(n+1)}(g|t) \mid N^{(n)}(\cdot|t)]\right] = \mathbb{E}\left[N^{(n)}\Big(\mu(g|\cdot) \;\Big|\; t\Big)\right] \;.$$

Hence, we obtain, for any $n \geq 1$,

$$\mathbb{E}\left[N^{(n)}(g|t)\right] = \mu^{\star n}(g|t) \;, \tag{5}$$

where $\mu^{\star n}$ is defined iteratively as follows: for any g,

$$\begin{cases} \mu^{\star 0}(g|t) &= g(t) \\ \mu^{\star (n+1)}(g|t) &= \mu\Big(\mu^{\star n}(g|\cdot) \;\Big|\; t\Big), \quad \text{for any } n \geq 0 \;. \end{cases} \tag{6}$$

We also note that the intensity measure of a component generated at center point t reads

$$M_1(\cdot|t) = \mathbb{E}[N(\cdot|t)] = \sum_{n\geq 0} \mu^{\star n}(\cdot|t) \;. \tag{7}$$

It is easy to see that if $\mu(\cdot|t) = \mu \circ S^t$, then $\mu^{\star n}(\cdot|t) = \mu^{\star n} \circ S^t$, where $\mu^{\star n}$ now denotes the standard convolution of measures. Then (7) with $t = 0$ corresponds to the formula given for $M_1(A|0)$ in [3, Page 184].

From (7), we deduce that, for any non-negative function g,

$$M_1(g|t) = g(t) + \sum_{n\geq 1} \mu^{\star n}(g|t)$$

and we conclude that

$$M_1(g) = \mathbb{E}[N(g)] = \int M_1(g|t)\, \mu_c(\mathrm{d}t) \;. \tag{8}$$

Note however that at this point, N so defined may not have locally finite intensity measure ($\mathbb{E}[N(g)]$ may be infinite for g bounded with compact support). This can be guaranteed by the following result.

Theorem 1. *Suppose that*

$$\zeta_1 := \sup_{t\in\mathbb{R}} \mu\left(\mathbb{R}^\ell | t\right) < 1 \; . \tag{9}$$

Then the component process $N(\cdot|t)$ defined by (4) has finite moment measure satisfying

$$M_1(\mathbb{R}^\ell|t) = \mathbb{E}[N(\mathbb{R}^\ell|t)] \leq \frac{1}{1-\zeta_1} \; .$$

Proof. Observe that, for any non-negative function g, we have by (6) that, for all $n \geq 0$ and all $t \in \mathbb{R}^\ell$,

$$\mu^{\star(n+1)}(g|t) \leq \zeta_1 \sup\left(\mu^{\star(n)}(g|\cdot)\right) \; ,$$

where $\sup(g)$ denotes the sup of the function g over $\mathbb{R}^\ell$. By induction, we get that

$$\sup\left(\mu^{\star(n)}(g|\cdot)\right) \leq \zeta_1^n \sup(g) \; ,$$

and the proof is concluded by applying (7). □

Consequently, applying (8), we conclude that under Condition (9), if μ_c is locally finite, then N admits a locally finite intensity measure. Note also that Condition (9) corresponds to the usual condition in the stationary case, see [3, Example 6.3(c)].

2.4. Density assumption

We now assume that the intensity measures μ_c and $\mu(\cdot|t)$ admit densities with respect to the Lebesgue measure on $\mathbb{R}^\ell$. We denote by λ_c the density of μ_c and by $d(\cdot - t; t)$ the density of $\mu(\cdot|t)$. In this notation the fact that $t \mapsto \mu(\cdot|t)$ is not shift invariant is apparent in the fact that $d(s-t;t)$ does not depend on $s-t$ only but also on t. Note also that the function $d(s-t;t)$ can be equivalently rewritten as a function of $s-t$ and s (using an obvious change of variable), which we do by introducing the function $p(\cdot;\cdot)$ defined on $\mathbb{R}^{\ell^2}$ by setting

$$p(s-t;s) = d(s-t;t), \quad \text{for all } s,t \in \mathbb{R}^\ell \; .$$

When localizing the non-stationary behavior, it will turn out to be more convenient to use the description with the density p, that we call the (non-stationary) fertility function, rather than with d. The intuitive reason is the following: coarsely speaking, $d(s-t;t)\,\mathrm{d}s = p(s-t;s)\,\mathrm{d}s$ represents the probability that a point t has an offspring occurring around location s in an elementary set $\mathrm{d}s$. From this view, the location s corresponds to the position where the probability mass is located and it is more convenient that the second argument corresponds to this location rather than the location t of the generating point.

Definition 1 (Non-stationary Hawkes process)**.** We say that the so defined non stationary Hawkes process has immigrant intensity function λ_c and varying fertility function $p(\cdot;\cdot)$. By Theorem 1 and (8), if

$$\zeta_1 = \sup_{t\in\mathbb{R}^\ell} \int p(s;t)\,\mathrm{d}s < 1 \quad \text{and} \quad |\lambda_c|_\infty < \infty \; , \tag{10}$$

then the point process admits a density function (the density of M_1) which is uniformly bounded by $|\lambda_c|_\infty/(1-\zeta_1)$.

The following argument will also be useful to simplify the proofs, since we often look at the behavior of N around a specific position t, which amounts to consider the behavior of $N \circ S^{-t}$ around the origin.

Remark 2. Let N be a non-stationary Hawkes process with center intensity λ_c and fertility function $p(\cdot;\cdot)$. For any $t \in \mathbb{R}$, the distribution of the shifted process $N \circ S^{-t}$ defined by $N \circ S^{-t}(g) = N(g(\cdot - t))$ for a function g, is that of a non-stationary Hawkes process with center intensity $\lambda_c(\cdot + t)$ and fertility function $p(\cdot; \cdot + t)$.

2.5. Locally stationary Hawkes processes

The non-stationary Hawkes processes under the density assumption, can still evolve quite arbitrarily in the space, as the functional parameters λ_c and $p(\cdot;\cdot)$ can be quite general. The stationary case corresponds to the case where λ_c is constant and $p(\cdot;\cdot)$ is constant over its second argument. This can be interpreted as a particular set of parameters for λ_c and $p(\cdot;\cdot)$, which we explicitly exhibit by introducing the following notation. In the stationary case, the immigrant intensity λ_c and fertility function $p(\cdot;\cdot)$ only depend on the constant $\lambda_c^{<\mathrm{S}>}$ and the function $p^{<\mathrm{S}>} : \mathbb{R}^\ell \to \mathbb{R}_+$ by setting

$$\lambda_c(t) =: \lambda_c^{<\mathrm{S}>}, \quad \text{for all } t \in \mathbb{R} \tag{11}$$

$$p(s;t) = p^{<\mathrm{S}>}(s), \quad \text{for all } t \in \mathbb{R} \text{ and } s \geq 0 \,. \tag{12}$$

We now wish to define a model of point process that can be locally interpreted as a stationary Hawkes process, in the same fashion as *locally stationary* autoregressive processes in time series (see [16]). The model is a doubly indexed point process $N_T(A)$, $A \in \mathbb{B}(\mathbb{R})$ such that for each $T > 0$, N_T is a non-stationary Hawkes process defined as previously. Here T correspond to the size of the observation window so that we only observe $N_T(A)$ for Borel sets $A \subseteq T\mathbb{D}$, where $\mathbb{D}$ is a fixed domain and $T\mathbb{D} = \{Tx,\, x \in \mathbb{D}\}$. The collection $(N_T)_{T>0}$ of non-stationary Hawkes processes are defined using the same μ_c and $t \mapsto \mu(\cdot|t)$ but scaled differently so that, if the observation window has the form $T\mathbb{D}$, then it matches the corresponding fixed domain $\mathbb{D}$ for these parameters. In this way, while the observations evolve in $T\mathbb{D}$ the parameter of interest is defined independently of T on the domain $\mathbb{D}$. We call this model a *locally stationary* Hawkes process and denote the fixed parameters by $\lambda_c^{<\mathrm{LS}>}$ and $p^{<\mathrm{LS}>}(\cdot;\cdot)$. For $\ell = 1$, as for the locally stationary time series, one typically takes $\mathbb{D} = [0,1]$.

Definition 2 (Locally stationary Hawkes process)**.** A locally stationary Hawkes process with *local immigrant intensity* $\lambda_c^{<\mathrm{LS}>}$ and *local fertility function* $p^{<\mathrm{LS}>}(\cdot;\cdot)$ is a collection $(N_T)_{T>0}$ of non-stationary Hawkes processes with respective immigrant intensity and fertility function given by $\lambda_{cT}(t) = \lambda_c^{<\mathrm{LS}>}(t/T)$ and varying fertility function given by $p_T(\cdot;t) = p^{<\mathrm{LS}>}(\cdot;t/T)$.

For a given *real location* t, the scaled location t/T is typically called an *absolute location* in $\mathbb{D}$ and denoted by u or v.

As explained in Definition 1, the following assumption, which corresponds to (10), guarantees that, for all $T > 0$, the non-stationary Hawkes process N_T admits a uniformly bounded intensity function.

(LS-1) We have

$$\zeta_1^{<\mathrm{LS}>} := \sup_{u \in \mathbb{R}^\ell} \int p^{<\mathrm{LS}>}(t;u)\,\mathrm{d}t < 1 \quad \text{and} \quad \left|\lambda_c^{<\mathrm{LS}>}\right|_\infty < \infty \,. \tag{13}$$

Under this assumption, moreover, for each absolute location $u \in \mathbb{R}^\ell$, the function $t \mapsto p^{<\mathrm{LS}>}(t;u)$ satisfies the required condition for the fertility function of a stationary Hawkes process. In the following, under (LS-1), for any absolute location u, we denote by $N(\cdot;u)$ a stationary Hawkes process with immigrant intensity $\lambda_c^{<\mathrm{LS}>}(u)$ and fertility function $t \mapsto p^{<\mathrm{LS}>}(t;u)$.

3. Main results

3.1. Local approximation of the log Laplace functional

An important tool for statistical applications is to have a local approximation of N_T as $T \to \infty$. Let us precise what we mean by "local" here. Let a fixed absolute location $u \in \mathbb{R}^\ell$ be given. Then N_T shifted at the real location Tu, namely $N_T \circ S^{-Tu}$ approximately follows the distribution of a stationary Hawkes process with intensity $\lambda^{<\mathrm{S}>} := \lambda_c^{<\mathrm{LS}>}(u)$ and fertility function $p^{<\mathrm{S}>} := p^{<\mathrm{LS}>}(\cdot;u)$. To this aim the following remark will be useful.

Remark 3. By Remark 2, the exact distribution of $N_T \circ S^{-Tu}$ can be obtained by replacing $\lambda_c^{<\mathrm{LS}>}(t/T)$ with $\lambda_c^{<\mathrm{LS}>}((t+Tu)/T) = \lambda_c^{<\mathrm{LS}>}(u+t/T)$ and $p^{<\mathrm{LS}>}(r;t/T)$ with $p^{<\mathrm{LS}>}(r;(t+Tu)/T) = p^{<\mathrm{LS}>}(r;u+t/T)$. In other words, $\lambda_c^{<\mathrm{LS}>}(v)$ is replaced by $\lambda_c^{<\mathrm{LS}>}(u+v)$ and $p^{<\mathrm{LS}>}(s;v)$ by $p^{<\mathrm{LS}>}(s;u+v)$.

We examine local approximations of the distribution of the locally stationary Hawkes process $(N_T)_{T>0}$ through the Laplace functional which is an efficient tool to describe the distribution of point processes. We denote the Laplace functional of N_T by

$$\mathcal{L}_T(g) = \mathbb{E}\left[\exp N_T(g)\right] = \mathbb{E}\left[\exp \int N_T(g|t)\; N_{cT}(\mathrm{d}t)\right] ,$$

where N_{cT} and $N_T(\cdot|t)$ are the corresponding center process and component process generated by a center at location t, respectively. Our goal is to derive the asymptotic behavior of $\mathcal{L}_T(S^{-Tu}g)$ as $T \to \infty$ for any given absolute location u and any function g. Under appropriate condition, it should converge to the Laplace functional applied on g of a stationary Hawkes process with immigrant constant intensity given by $\lambda_c^{<\mathrm{LS}>}(u)$ and with fertility function given by $p^{<\mathrm{LS}>}(\cdot;u)$. It is in fact more interesting to investigate convergence of the log-Laplace functional using the norm $|\cdot|_{\mathcal{O},K}$ defined in Section 2.1 by authorizing g to depend on an auxiliary variable $z \in U$. We will, by convenient abuse of notation, continue to write $\mathcal{L}(g)$ in this setting, to denote the function $z \mapsto \mathcal{L}(g(\cdot,z))$ defined on U. Therefore using the notation $\bar{S}$ introduced in Section 2.1, we now investigate the behavior, as $T \to \infty$, for any given $u \in \mathbb{R}^\ell$, of

$$\mathcal{L}_T(\bar{S}^{-Tu}g) : z \mapsto \mathbb{E}\left[\exp N_T(g(\cdot - Tu, z))\right] ,$$

seen as a function defined on $z \in U$. An example of application is to obtain approximations of cumulants of arbitrary orders, since they can be obtained as

$$\mathrm{Cum}\left(N(g_1),\dots,N(g_m)\right) = \partial^{1_m}|_{z=0_m} \log \mathcal{L}(z_1 g_1 + \dots + z_m g_m) , \tag{14}$$

where 1_m and 0_m denote the m-dimensional vectors filled with ones and zeros, respectively. We develop this idea in Section 3.2.

The following assumptions use some of the norms introduced in Section 2.1.

(LS-2) We have $|\lambda_c^{<\mathrm{LS}>}|_\infty < \infty$ and

$$\xi_c^{(\beta)} := \sup_{u \neq v} \frac{|\lambda_c^{<\mathrm{LS}>}(v) - \lambda_c^{<\mathrm{LS}>}(u)|}{|v-u|^\beta} < \infty .$$

(LS-3) We have $\left|\xi^{(\beta)}\right|_1 < \infty$, where

$$\xi^{(\beta)}(r) := \sup_{u \neq v} \frac{|p^{<\mathrm{LS}>}(r;v) - p^{<\mathrm{LS}>}(r;u)|}{|v-u|^\beta} \;.$$

(LS-4) We have

$$\zeta_\infty^{<\mathrm{LS}>} := \sup_{u\in\mathbb{R}} \left|p^{<\mathrm{LS}>}(.;u)\right|_\infty < \infty \;, \tag{15}$$

$$\zeta_{(\beta)}^{<\mathrm{LS}>} = \sup_{u\in\mathbb{R}} \left|p^{<\mathrm{LS}>}(\cdot;u)\right|_{(\beta)} < \infty \;. \tag{16}$$

These assumptions can be interpreted as smoothness conditions on $\lambda_c^{<\mathrm{LS}>}$ ((LS-2)) and on $p^{<\mathrm{LS}>}(\cdot;\cdot)$ with respect to its second argument ((LS-3)) and some uniform decreasing condition on $p^{<\mathrm{LS}>}(\cdot;\cdot)$ with respect to its first argument ((LS-4)).

We can now state the main result, where the appearing norms $|\cdot|_{\mathcal{O},K}$, $|\cdot|_{\bar{\mathcal{O}},K,q}$, $|\cdot|_{\bar{\mathcal{O}},K,(\beta)}$ and the sets $B_{\bar{\mathcal{O}}}(R;K,q)$ are all defined in Section 2.1. In this theorem, for any $u \in \mathbb{R}^\ell$, we denote by $\mathcal{L}(\cdot;u)$ the Laplace functional of the stationary Hawkes process with constant intensity $\lambda_c^{<\mathrm{LS}>}(u)$ and fertility function $p^{<\mathrm{LS}>}(\cdot;u)$.

Theorem 2. *Let $\beta \in (0,1]$. Assume (LS-1), (LS-2), (LS-3), (LS-4). Let $g \in \bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$ such that for all compact set $K \subset U$,*

$$|g|_{\bar{\mathcal{O}},K,1} < \left(-\frac{1}{2}\log\zeta_1^{<LS>}\right)(\zeta_1^{<LS>})^{1/2}(\zeta_\infty^{<LS>})^{-1}(1-\zeta_1^{<LS>})^{1/2} \;, \tag{17}$$

$$|g|_{\bar{\mathcal{O}},K,\infty} < -\frac{1}{2}\log\zeta_1^{<LS>} - (\zeta_1^{<LS>})^{-1/2}(\zeta_\infty^{<LS>})(1-\zeta_1^{<LS>})^{-1/2}\,|g|_{\bar{\mathcal{O}},K,1} \;. \tag{18}$$

Then for each $T>0$ and each $u \in \mathbb{R}^\ell$, $z \mapsto \mathcal{L}_T(g(\cdot,z))$ and $z \mapsto \mathcal{L}(g(\cdot,z);u)$ can be expressed as

$$\mathcal{L}_T(g) = \exp\circ\mathcal{K}_T(g) \quad \text{and} \quad \mathcal{L}(g;u) = \exp\circ\mathcal{K}(g;u) \;,$$

where $\mathcal{K}_T(g)$ and $\mathcal{K}(g;u)$ are holomorphic functions on U. Moreover, for all $T>0$, $u\in\mathbb{R}^\ell$ and all compact sets $K \subset U$,

$$\left|\mathcal{K}_T(\bar{S}^{-Tu}g) - \mathcal{K}(g;u)\right|_{\mathcal{O},K} \le C_1\left(|g|_{\bar{\mathcal{O}},K,(\beta)} + C_2\,|g|_{\bar{\mathcal{O}},K,1}\right)T^{-\beta} \;, \tag{19}$$

where

$$C_1 = \frac{\left|\xi^{(\beta)}\right|_1 |\lambda_c^{<LS>}|_\infty}{\left((\zeta_1^{<LS>})^{1/2} - \zeta_1^{<LS>}\right)^2} + \frac{\xi_c^{(\beta)}}{(\zeta_1^{<LS>})^{1/2} - \zeta_1^{<LS>}} \quad \text{and} \quad C_2 = \frac{\zeta_{(\beta)}^{<LS>}}{(\zeta_1^{<LS>})^{1/2} - \zeta_1^{<LS>}} \tag{20}$$

Proof. This result requires preliminary results to be found in Sections 6.1 (about the derivation of the log-Laplace functional for non-stationary Hawkes processes) and 6.2 (about local approximations for log-Laplace functional of the component processes $N_T(\cdot|t)$). The proof is then completed in Section 6.3. □

Remark 4. Since we assume $g \in \bar{\mathcal{O}}_1(U)$ in Theorem 2, we know that $|g|_{\bar{\mathcal{O}},K,1} < \infty$ in the right-hand side of (19). However the assumptions on g do not guarantee that $|g|_{\bar{\mathcal{O}},K,(\beta)} < \infty$. This condition needs to be verified in order to apply (19) meaningfully, this fact should be checked first.

This theorem shows that for T large, the Laplace functional of the non-stationary Hawkes process N_T translated at location Tu can be approximated by that of the stationary Hawkes process $N(\cdot;u)$. It moreover provides in (19) a rate of convergence $T^{-\beta}$ of this approximation in an adequate norm. Since the Laplace function characterizes the distribution of Point process, it is not surprising that an immediate corollary of Theorem 2 is that N_T translated at location Tu converges in distribution to $N(\cdot;u)$ as $T\to\infty$. Recall that the set of locally finite nonnegative Borel measures on $\mathbb{R}^\ell$ endowed with the usual weak convergence of locally finite measures can be equipped with a metric to constitute a complete separable metric space, see [3, Theorem A2.6.III].

Corollary 3. *Let $\beta\in(0,1]$. Assume (LS-1), (LS-2), (LS-3), (LS-4). Then, for any $u\in\mathbb{R}^\ell$, as $T\to\infty$, the point process $N_T\circ S^{-Tu}$ converges in distribution to $N(\cdot;u)$.*

Proof. By [18, Proposition 11.1.VIII], it is sufficient to show that, for a given continuous and compactly supported function $h:\mathbb{R}^\ell\to\mathbb{R}$, the random variable $N_T(S^{-Tu}h)$ converges in distribution to $N(h;u)$. Let us define, for all $(t,z)\in\mathbb{R}^\ell\times\mathbb{C}$, $g(t,z)=z\,g(t)$. Let U be the open ball of $\mathbb{C}$ with center 0 and radius $r>0$. Then for any $q\in[1,\infty]$ and any compact set $K\subset U$, we have $|g|_{\bar{\mathcal{O}},K,q}\leq r\,|h|_q$, and similarly, $|g|_{\bar{\mathcal{O}},K,(\beta)}\leq r\;|h|_{(\beta)}$. Since $|h|_q$ and $|h|_{(\beta)}$ are finite, we conclude that g satisfies (17) and (18) for r small enough and that $|g|_{\bar{\mathcal{O}},K,(\beta)}<\infty$. Thus Theorem 2 gives that for $r>0$ small enough, we have that $z\mapsto\mathbb{E}[\exp(z\,N_T(S^{-Tu}h))]$ and $z\mapsto\mathbb{E}[\exp(z\,N(h;u))]$ are holomorphic on U and the former converges uniformly to the latter. This is enough to insure the convergence in distribution of $N_T(S^{-Tu}h)$ to $N(h;u)$. ☐

Observe that in Corollary 3, we do not exploit the rate of convergence $T^{-\beta}$ established in Theorem 2. Approximations on the cumulants will be more precise in that respect.

3.2. Local approximation of the cumulants

Recall that the cumulant of any order can be obtained from the log-Laplace functional through Equation (14), which is valid whenever $g_1,\ldots,g_m$ satisfy $\mathbb{E}[|N(g_j)|^m]<\infty$. Using Theorem 2, we obtain the following result for approximating the cumulants of N_T translated at location Tu with those of $N(\cdot;u)$.

Theorem 4. *Let $\beta\in(0,1]$. Assume (LS-1), (LS-2), (LS-3), (LS-4). Then, for any T and any $u\in\mathbb{R}^\ell$ and all bounded integrable functions $h:\mathbb{R}^\ell\to\mathbb{R}$, the random variables $N_T(h)$ and $N(h;u)$ admit finite exponential moments, that is, there exists $a>0$ such that $\mathbb{E}[\exp(a\,|N_T(h)|)]$ and $\mathbb{E}[\exp(a\,|N(h;u)|)]$ are finite. Let moreover for any $m\geq1$, $g_1,\ldots,g_m$ be real valued bounded integrable functions on $\mathbb{R}^\ell$. Then for any T and any $u\in\mathbb{R}^\ell$, we have*

$$\left|\mathrm{Cum}\left(N_T(S^{-Tu}g_1),\ldots,N_T(S^{-Tu}g_m)\right)-\mathrm{Cum}\left(N(g_1;u),\ldots,N(g_m;u)\right)\right|$$
$$\leq\frac{2^{m-1}C_1\,T^{-\beta}}{(-\log\zeta_1^{<LS>})^{m-1}}\left\{\sum_{j=1,\ldots,m}\left(|g_j|_{(\beta)}+C_2\,|g_j|_1\right)\right\}\left\{\sum_{j=1,\ldots,m}\left(|g_j|_\infty+C_3\,|g_j|_1\right)\right\}^{m-1},$$

where C_1 and C_2 are defined in (20), and

$$C_3=\frac{\zeta_\infty^{<LS>}}{(\zeta_1^{<LS>})^{1/2}(1-\zeta_1^{<LS>})^{1/2}}\,. \tag{21}$$

Proof. The proof of this result is given in Section 6.4. ☐

3.3. Local mean density

Applying Theorem 4 with $m = 1$, we obtain that the intensity measure M_{1T} of the non-stationary point process N_T can be approximated by the intensity measure $M_1^{<\mathrm{LS}>}(\cdot; u)$ of the stationary Hawkes process $N(\cdot; u)$, namely for any bounded and integrable function g defined on $\mathbb{R}^\ell$, we have

$$\left|M_{1T}(S^{-Tu} g) - M_1^{<\mathrm{LS}>}(g; u)\right| \le C \, \left(|g|_{(\beta)} + |g|_1\right) \, T^{-\beta} \, ,$$

where C is a positive constant. This result can be stated in a handier way by using the densities of M_{1T} and $M_1^{<\mathrm{LS}>}(\cdot; u)$. As seen in Definition 1, for all $T > 0$, M_{1T} admits a uniformly bounded density, hereafter denoted by m_{1T}. Since $N(\cdot; u)$ is a stationary Hawkes process, we know from [3, Eq. (6.3.26) in Example 6.3(c)] that $M_1^{<\mathrm{LS}>}(\cdot; u)$ admits a constant mean density

$$m_1^{<\mathrm{LS}>}(u) = \frac{\lambda_c^{<\mathrm{LS}>}(u)}{1 - \int p^{<\mathrm{LS}>}(\cdot; u)} \, . \tag{22}$$

We call $m_1^{<\mathrm{LS}>}(u)$ the *local mean density* at absolute location u. We have the following result.

Corollary 5. *Let $\beta \in (0, 1]$. Assume (LS-1), (LS-2), (LS-3), (LS-4). Then, for any T, N_T admits a density function m_{1T} satisfying*

$$|m_{1T}|_\infty \le \frac{|\lambda_c^{<LS>}|_\infty}{\ell - \zeta_1^{<LS>}} \, .$$

Moreover, we have, for all $u \in \mathbb{R}^\ell$, $T > 0$ and $b > 0$,

$$\operatorname*{ess\,sup}_{t \,:\, |t - Tu| \le b} \left|m_{1T}(t) - m_1^{<LS>}(u)\right| \le C_1 \, \left(C_2 + b^\beta\right) \, T^{-\beta} \, , \tag{23}$$

where $m_1^{<LS>}(u)$ is defined in (22), and C_1 and C_2 are defined in (20).

Proof. The existence and uniform boundedness of m_{1T} is embedded in Definition 1. Let now $u \in \mathbb{R}^\ell$, $T > 0$ and $b > 0$. Applying Theorem 4 with $m = 1$, we have for all bounded and integrable functions g defined on $\mathbb{R}^\ell$,

$$\left|\int g(t - Tu) m_{1,T}(t) \, \mathrm{d}t - \frac{\lambda_c(u)}{1 - \int p^{<\mathrm{LS}>}(\cdot; u)} \int g\right| \le C_1 \, \left(|g|_{(\beta)} + C_2 \, |g|_1\right) \, T^{-\beta} \, .$$

We define the function f on $\mathbb{R}^\ell$ by

$$f(t) = m_{1T}(t) - \frac{\lambda_c(u)}{1 - \int p^{<\mathrm{LS}>}(\cdot; u)} \, ,$$

so that the previous display reads

$$\left|\int g(t - Tu) f(t) \mathrm{d}t\right| \le C_1 \, \left(|g|_{(\beta)} + C_2 \, |g|_1\right) \, T^{-\beta} \, . \tag{24}$$

Let a be any positive number strictly smaller than the left-hand side of (23), that is, $a < \operatorname{ess\,sup}_{|t - Tu| \le b} |f(t)|$. Then there exists a Borel set $A \subset \{t \; : \; |t - Tu| \le b\}$ with positive Lebesgue measure, $\int 1_A > 0$, such that $|f(t)| \ge a$ for all $t \in A$. Let g be the function defined so that $g(t-Tu)$ is equal to the sign of $f(t)$ if $t \in A$ and to zero everywhere else. Then we get that

$$a \int 1_A \leq \int_A |f| = \left| \int g(t-Tu) f(t) \mathrm{d}t \right|$$

On the other hand we have $|g|_1 = \int 1_A$ and

$$|g|_{(\beta)} = \int |g(s)|\, |s|^\beta \, \mathrm{d}t \leq \int_A |t-Tu|^\beta \, \mathrm{d}t \leq b^\beta \int 1_A \; ,$$

where we used that $A \subset \{t \; : \; |t-Tu| \leq b\}$. Inserting these bounds in (24) gives that

$$a \int 1_A \leq C_1 \left(b^\beta \int 1_A + C_2 \int 1_A \right) \; T^{-\beta} \; .$$

Simplifying by $\int 1_A > 0$ and letting a tend to $\operatorname{ess\,sup}_{|t-Tu| \leq b} |f(t)|$, we get the result. $\square$

4. Time-frequency analysis of point processes

One of the benefits of locally stationary time series is that they provide a non-parametric statistical framework for time frequency analysis of time series, see [19] for a recent contribution. We show here how such ideas can be applied to locally stationary processes. Throughout this section, we take $\ell = 1$ for sake of convenience and $\mathbb{D} = [0, 1]$. Most of the definitions and results easily extend to $\ell \geq 2$.

4.1. Local Bartlett spectrum

Following [3, Proposition 8.2.I], the Bartlett spectrum Γ of a second order stationary point process N on $\mathbb{R}$ is defined as the (unique) non-negative measure on $\mathbb{R}$ such that, for any bounded and compactly supported function f on $\mathbb{R}$,

$$\mathrm{Var}\big(N(f)\big) = \Gamma(|\hat{f}|^2) = \int \left| \hat{f}(\omega) \right|^2 \; \Gamma(\mathrm{d}\omega) \; ,$$

where $\hat{f}$ denotes the Fourier transform of f,

$$\hat{f}(\omega) = \int f(t) \; \mathrm{e}^{-\mathrm{i}t\omega} \; \mathrm{d}t \; .$$

For stationary Hawkes processes with immigrant intensity λ_c and fertility function p, the Bartlett spectrum admits a density given by

$$\Gamma(\mathrm{d}\omega) = \frac{\lambda_c}{2\pi(1 - \int p)} \; |1 - \hat{p}(\omega)|^{-2} \; \mathrm{d}\omega \; ,$$

see [3, Example 8.2(e)]. Under (LS-1), applying this result to the stationary Hawkes process $N(\cdot; u)$, we have, for any bounded and compactly supported function f,

$$\mathrm{Var}\big(N(f; u)\big) = \Gamma^{<\mathrm{LS}>} \left(|\hat{f}|^2; u \right) \; , \tag{25}$$

where

$$\Gamma^{<\mathrm{LS}>}(\mathrm{d}\omega; u) = \frac{\lambda_c^{<\mathrm{LS}>}(u)}{2\pi(1 - \int p^{<\mathrm{LS}>}(\cdot; u))} \; \left| 1 - \hat{p}^{<\mathrm{LS}>}(\omega; u) \right|^{-2} \; \mathrm{d}\omega \; , \tag{26}$$

with

$$\hat{p}^{<\mathrm{LS}>}(\omega;u) = \int p^{<\mathrm{LS}>}(t;u)\, \mathrm{e}^{-\mathrm{i}t\omega}\, \mathrm{d}t \,.$$

We call $\Gamma^{<\mathrm{LS}>}(\cdot;u)$ the *local Bartlett spectrum* at absolute location u. We have the following result, which says that, although N_T is not stationary, for T large enough, its variance in the neighborhood of Tu can be approximated by using the local Bartlett spectrum at absolute location u.

Corollary 6. *Let $\beta \in (0,1]$. Assume (LS-1), (LS-2), (LS-3), (LS-4). Then, for all $u \in \mathbb{R}$, $T > 0$, and all bounded functions f supported inside $[-b,b]$ for some $b > 0$, we have*

$$\left|\mathrm{Var}\left(N_T(S^{-Tu}f)\right) - \Gamma^{<LS>}(|\hat{f}|^2;u)\right| \leq \frac{8\,C_1\,(b^\beta + C_2)}{-\log \zeta_1^{<LS>}}\, |f|_1\, \left(|f|_\infty + C_3\,|f|_1\right) T^{-\beta} \,, \qquad (27)$$

where $\Gamma^{<LS>}(\cdot;u)$ is defined in (26), C_1 and C_2 are defined in (20), and C_3 is defined in (21).

Proof. Let $u \in \mathbb{R}$, $T > 0$ and f be a bounded and compactly supported function. Applying Theorem 4 with $g_1 = g_2 = f$ and (25), we get that

$$\left|\mathrm{Var}\left(N_T(S^{-Tu}f)\right) - \Gamma^{<\mathrm{LS}>}(|\hat{f}|^2;u)\right| \leq \frac{8\,C_1\,T^{-\beta}}{-\log \zeta_1^{<\mathrm{LS}>}} \left(|f|_{(\beta)} + C_2\,|f|_1\right) \left(|f|_\infty + C_3\,|f|_1\right) \,.$$

To conclude the proof we observe that if f is supported inside $[-b,b]$, then $|f|_{(\beta)} \leq b^\beta\,|f|_1$. ☐

4.2. Kernel estimation of the local Bartlett spectrum

Let f be a test function and m a moment function (such as $m(x) = x$, $m(x) = x^2$, ...). Let b_1 be a given time bandwidth and u_0 a fixed time in $[0;1]$ (namely, $u_0 = t_0/T$ with $t_0 \in [0;T]$). We build an estimator of $\mathbb{E}[m(N(f;u_0))]$ based on the empirical observations of N_T and defined by

$$\widehat{E}[m \circ N_T(f);w] := \frac{1}{T}\int m \circ N_T(f(\cdot - t))w(t/T)\, \mathrm{d}t,$$

where w denotes a weight function: $w = W_{b_1,u_0} : u \mapsto b_1^{-1}W((u-u_0)/b_1)$ for some fixed kernel function W. In practice, f should be compactly supported, so that this integral can be computed from a finite set of observations in $[0,T]$. Let K be a real valued kernel compactly supported and its Fourier transform $\hat{K}$ such that $\int |\hat{K}(\omega)|^2\, \mathrm{d}\omega = 1$. Let b_2 be a given frequency bandwidth and ω_0 a fixed frequency. We wish to estimate the quantity

$$\gamma_{b_2}(\omega_0;u_0) := \int \frac{1}{b_2}|\hat{K}((\omega-\omega_0)/b_2)|^2\, \Gamma^{<\mathrm{LS}>}(\mathrm{d}\omega;u_0), \qquad (28)$$

which in turn is an approximation of the density of $\Gamma^{<\mathrm{LS}>}(\cdot;u_0)$ at ω_0 when it exists. We denote by $f = K_{b_2,\omega_0}$ the kernel having Fourier transform $\omega \mapsto b_2^{-1/2}\hat{K}((\omega-\omega_0)/b_2)$. Consequently, by inverse Fourier transform, we get that $K_{b_2,\omega_0}(t) = b_2^{1/2}\mathrm{e}^{\mathrm{i}\omega_0 t}K(b_2 t)$. Finally, we take $m(x) = x^2$ and $m(x) = x$ successively to define

$$\widehat{\gamma}_{b_2,b_1}(\omega_0;u_0) = \widehat{E}\left(|N_T(K_{b_2,\omega_0})|^2;W_{b_1,u_0}\right) - \left|\widehat{E}\left(N_T(K_{b_2,\omega_0});W_{b_1,u_0}\right)\right|^2. \qquad (29)$$

The quantity $\widehat{\gamma}_{b_2,b_1}(\omega_0;u_0)$ is a natural estimator of $\mathrm{Var}\big(N_T(S^{-Tu_0}K_{b_2,\omega_0})\big)$. Thus, by (25), (28) and Corollary 6, $\widehat{\gamma}_{b_2,b_1}(\omega_0;u_0)$ is a sensible estimator of $\gamma_{b_2}(\omega_0;u_0)$.

5. Numerical experiments

5.1. Simulation of locally stationary Hawkes processes

Following Definition 2, we consider a locally stationary Hawkes process $(N_T)_{T>0}$ with local immigrant intensity $\lambda_c^{<\mathrm{LS}>}$ and local fertility function $p^{<\mathrm{LS}>}(\cdot;\cdot)$. Provided that $s \mapsto p^{<\mathrm{LS}>}(s;u)$ is supported on the positive half line for all u, the conditional intensity of a stationary Hawkes process recalled in (1) can be extended to the non-stationary Hawkes processes N_T, namely

$$\lambda_T(t) := \lambda_c^{<\mathrm{LS}>}(t/T) + \sum_{t_i<t} p^{<\mathrm{LS}>}(t-t_i;t/T) \ ,$$

where $(t_i)_{i\in\mathbb{Z}}$ denote the points of N_T. It follows that, for a given $T>0$, the non-stationary Hawkes process N_T can be simulated over the interval $[0,T]$ by using Ogata's modified thinning algorithm (see [20]). This algorithm is a recursive algorithm which only requires that, having simulated the process up to time t, one is able to provide an upper bound

$$M(t) \geq \sup_{s\in[t;T]} \left(\lambda_c^{<\mathrm{LS}>}(s/T) + \sum_{t_i<t} p^{<\mathrm{LS}>}(s-t_i;s/T) \right) \ .$$

Choosing $\lambda_c^{<\mathrm{LS}>}$ and $p^{<\mathrm{LS}>}(\cdot;\cdot)$ adequately, one can for instance use the bound

$$M(t) = \sup_{u\in\mathbb{R}} \left(\lambda_c^{<\mathrm{LS}>}(u)\right) + \sum_{t_i<t} \sup_{u\in\mathbb{R}} \sup_{s>t} \left(p^{<\mathrm{LS}>}(s-t_i;u)\right) \ .$$

A classical drawback of Ogata's algorithm is that, in order to initiate the simulation, say at time $t=0$, one needs in principle to have the points $t_i<0$ at hand, which is of course not the case. In a stationary context, one can use a burn-in period and shift the time origin to start the process in a close to steady state. In a non-stationary context, one is in fact allowed to assume the process to be empty on the negative half line, which would correspond to have $\lambda_c^{<\mathrm{LS}>}(u)=0$ for all $u<0$. This setting makes Ogata's algorithm perfect to simulate N_T over $[0,T]$. However, to avoid border effects at the beginning of the sample, we used a burn-in period to initiate the process in a close to steady state, which corresponds to setting $\lambda_c^{<\mathrm{LS}>}(u)=\lambda_c^{<\mathrm{LS}>}(0)$ and $p^{<\mathrm{LS}>}(\cdot;u)=p^{<\mathrm{LS}>}(\cdot;0)$ for all $u<0$.

5.2. Examples

We consider a specific class of examples by taking a constant immigrant intensity λ_c and by focusing on a local fertility function with the shape of a Gamma distribution. Namely, for positive parameters δ, ζ, $\eta\geq 1$ and θ, let us denote by p_G the fertility function defined for all $s\in\mathbb{R}$ by

$$p_G(s;\delta,\zeta,\eta,\theta) = \zeta(s-\delta)^{\eta-1}\frac{\theta^\eta \mathrm{e}^{-\theta(s-\delta)}}{G(\eta)} 1_{s>\delta}$$

with $G(x)=\int_0^\infty s^{x-1}\mathrm{e}^{-s}\,\mathrm{d}s$ denoting the usual Gamma function. Note that δ is a time-shift parameter which induces a periodic phenomenon in the self-exciting generating process: each event may generate a new event only after a delay δ. For this specific fertility function, we can easily compute the quantities appearing in our assumptions (*e.g.* $\int p_G=\zeta$ and $p_G(s;\delta,\zeta,\eta,\theta)$ is maximal for $s=\frac{\eta-1}{\theta}+\delta$) and we can exactly compute the corresponding mean density $m_{G1}(\delta,\zeta,\eta,\theta)$ and Bartlett spectrum $\Gamma_G(\mathrm{d}\omega;\delta,\zeta,\eta,\theta)$:

- $m_{G1}(\delta,\zeta,\eta,\theta) = \dfrac{\lambda_c}{1-\zeta}$ and

- $\Gamma_G(\mathrm{d}\omega;\delta,\zeta,\eta,\theta) = \dfrac{m_{G1}(\delta,\zeta,\eta,\theta)}{2\pi|1-\hat{p}_G(\omega;\delta,\zeta,\eta,\theta)|^2}\,\mathrm{d}\omega$, with

$$\hat{p}_G(\omega;\delta,\zeta,\eta,\theta) = \zeta \mathrm{e}^{-\mathrm{i}\omega\delta}\left(1+\frac{\mathrm{i}\omega}{\theta}\right)^{-\eta} .$$

Now, letting the parameters depend on the real time u provides the definition of a local fertility function,

$$p^{<\mathrm{LS}>}(s;u) = p_G(s;\delta(u),\zeta(u),\eta(u),\theta(u)) .$$

The local mean density $m_1^{<\mathrm{LS}>}(u)$ and the local Bartlett spectrum $\Gamma^{<\mathrm{LS}>}(\cdot;u)$ can be defined accordingly from m_{G1} and Γ_G, respectively. In our examples, the shape parameter η remains constant and the other parameters are Lipschitz functions of u, assumed to be constant outside the interval $(0,1)$. Such a choice for the fertility function satisfies (LS-1), (LS-3) and (LS-4) with $\beta = 1$ provided that

$$\zeta_1^{<\mathrm{LS}>} = \sup_{u\in[0,1]} \zeta(u) < 1 , \quad \inf_{u\in[0,1]} \theta(u) > 0 ,$$

and, if δ is not constant, one has moreover to assume that $\eta \geq 2$. We focus our numerical study on two examples:

- *Example 1 [Exponential case without delay]:*

$$\lambda_c \equiv 0.5, \quad \delta \equiv 0, \quad \eta \equiv 1, \quad \zeta(u) = (\cos(2\pi u)+2)/4 \quad \text{and} \quad \theta(u) = \cos(2\pi u) + 3/2 \quad \text{for } u \in [0,1].$$

- *Example 2 [Gamma case with varying delay]:*

$$\lambda_c \equiv 0.5, \quad \eta \equiv 2, \quad \zeta \equiv 0.5, \quad \theta \equiv 1 \quad \text{and}$$
$$\delta(u) = (6-10u) \times 1_{[0;1/2]}(u) + (10u-4) \times 1_{(1/2;1]}(u) \ \text{ for } u \in [0,1].$$

Note that Example 1 has a time varying local mean density $m_1^{<\mathrm{LS}>}$ (since ζ varies) and Example 1 has a constant local mean density $m_1^{<\mathrm{LS}>}$. Both examples, however, exhibit time varying local Bartlett spectra $\Gamma^{<\mathrm{LS}>}$.

Figure 1 displays the theoretical local intensity $m_1^{<\mathrm{LS}>}$ (as a function of the absolute time $u \in [0,1]$) and the theoretical local Bartlett spectrum $\Gamma^{<\mathrm{LS}>}$ (as a function of the absolute time $u \in [0,1]$ and the frequency $\omega \in [0,1]$) for Example 1 and Figure 2 displays the theoretical local Bartlett spectrum $\Gamma^{<\mathrm{LS}>}$ (as a function of the absolute time $u \in [0,1]$ and the frequency $\omega \in [0,2]$) for Example 2. Because in the second example, the delay δ is varying linearly between 6 and 1 for u going from 0 to 1/2 and then back to 6 for $u \in [1/2,1]$, we see the spectral content evolving accordingly with a peak frequency evolving as the reciprocal of the delay (increasing for u going from 0 to 1/2 and then decreasing for $u \in [1/2,1]$). We can simulate one trajectory of N_T for each example over the interval $[0;T]$ by using Ogata's algorithm as described in Section 5.1.

Figure 3 and Figure 4 display the associated conditional intensities $\lambda_T(t)$ for $t \in [0,T]$ for these two simulated point processes with $T = 10000$. The fact that the mean density is varying in Example 1 is visible on Figure 3 as the conditional intensity sharply decreases in the middle of the sample. On the contrary, the conditional intensity is fluctuating around the same average in Figure 4 which matches the fact that the mean intensity is constant in this example.

Based on these two samples of N_T, we finally compute the estimator $\widehat{\gamma}_{b_2,b_1}(\omega;u)$ defined by (29), over an appropriate grid for $(\omega;u)$. We set $b_2 = 0.05$ and $b_1 = 0.1$ in these experiments and we used $[-1/2, 1/2]$-supported triangular shapes for kernels K and W. The obtained estimates of the local intensity and local Bartlett spectra for Example 1 and Example 2 are respectively given in Figure 5 and in Figure 6.

We observe that the estimated local Bartlett spectra show the main features of the true underlying spectra, which illustrates the approximation result derived in Corollary 6.

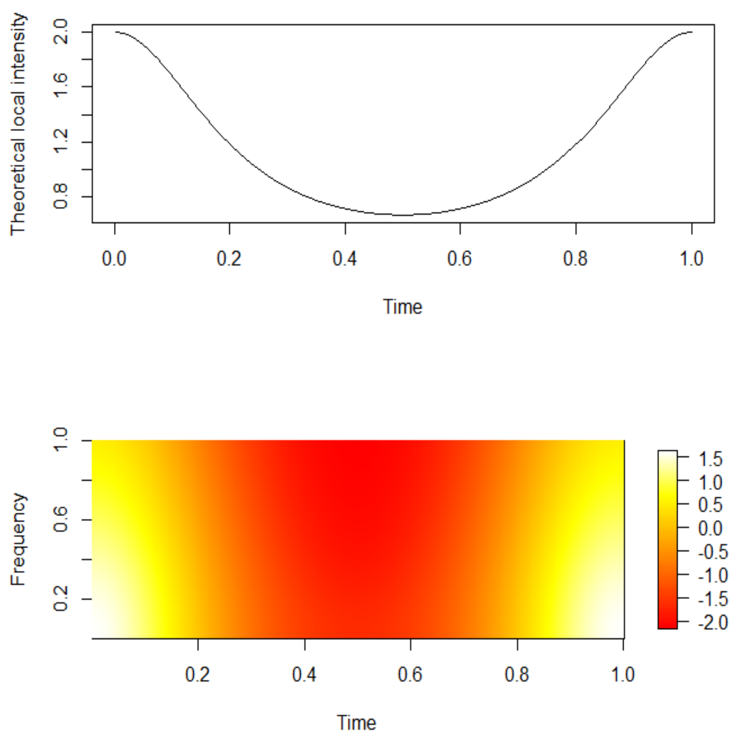


Figure 1: *Theoretical local intensity (top) and Bartlett spectrum (bottom) for Example 1.*

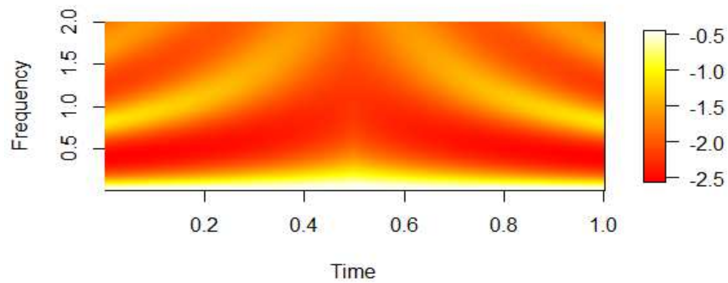


Figure 2: *Theoretical local Bartlett spectrum for Example 2.*

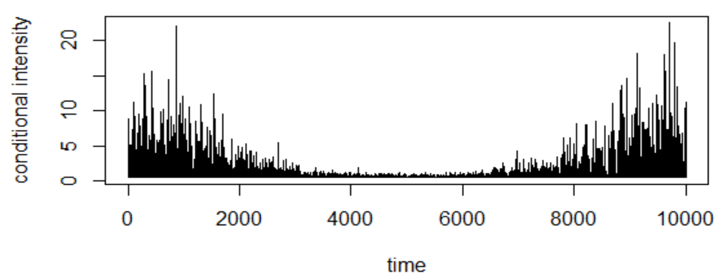


Figure 3: *Conditional intensity function of a simulated Hawkes process with respect to Example 1, with* $T = 10000$.

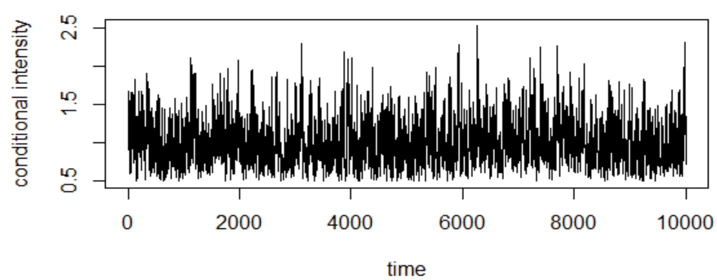
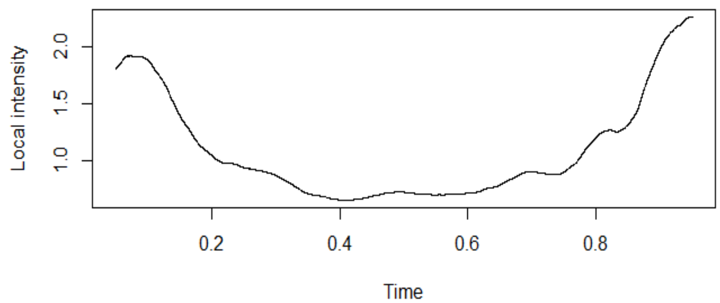


Figure 4: *Conditional intensity function of a simulated Hawkes process with respect to Example 2, with* $T = 10000$.



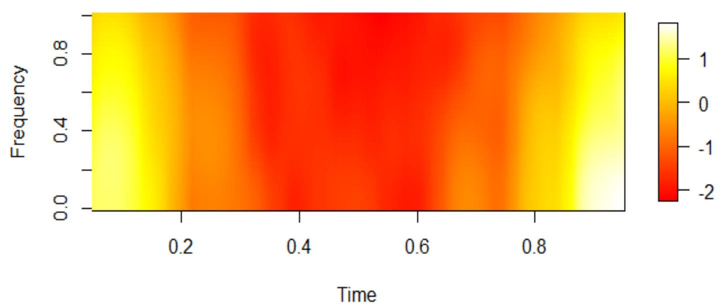


Figure 5: *Estimation of the local intensity (top) and of the local Bartlett spectrum (bottom) for Example 1.*

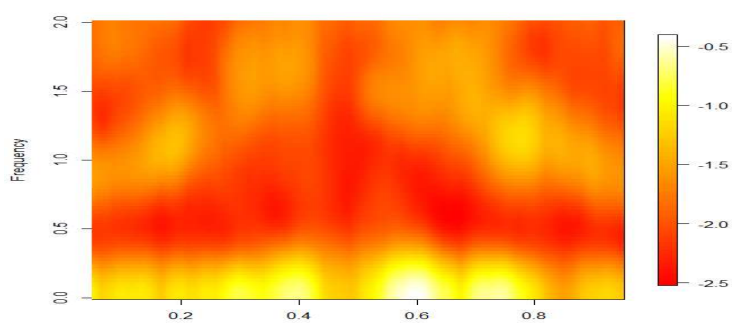


Figure 6: *Estimation of the local and Bartlett spectrum for Example 2.*

6. Proofs

6.1. Laplace functional of non-stationary Hawkes processes

In this section we suppose that N is a non-stationary Hawkes process as defined in Section 2.3 with immigrant intensity function λ_c and varying fertility function $p(\cdot;\cdot)$ satisfying (10).

We define

$$\mathcal{L}(g|t) = \mathbb{E}\left[\exp N(g|t)\right] \ , \tag{30}$$

conditioning on N_c, and using that N_c is a PPP with intensity λ_c, we get that, for well chosen functions g,

$$\mathcal{L}(g) = \mathbb{E}\left[\exp \int \log \mathcal{L}(g|t)\ N_c(\mathrm{d}t)\right] = \exp \int \left(\mathcal{L}(g|t) - 1\right)\ \lambda_c(t)\ \mathrm{d}t \ . \tag{31}$$

By (4) and monotone convergence we have, for all non-negative functions g,

$$\mathcal{L}(g|t) = \lim_{n\to\infty} \mathcal{L}_n(g|t) \ , \tag{32}$$

where

$$\mathcal{L}_n(g|t) = \mathbb{E}\left[\exp \sum_{k=0}^{n} N^{(k)}(g|t)\right] \ .$$

Moreover, by dominated convergence, Equation (32) remains valid for complex valued functions g, provided that $\mathcal{L}(|g|\,|\,t) < \infty$. Let us define, for functions g and h and $t \in \mathbb{R}^\ell$,

$$[\Phi_g(h)](t) = g(t) + \int \left(\mathrm{e}^{h(s)} - 1\right)\ p(s-t;s)\ \mathrm{d}s \ . \tag{33}$$

The integral in (33) is always defined if h is non-negative but may not be finite. If h is complex-valued, $\Phi_g(h)$ is well defined whenever $\Phi_g(|h|) < \infty$. We denote the n-th composition of the operator Φ_g by

$$\Phi_g^n = \underbrace{\Phi_g \circ \cdots \circ \Phi_g}_{n \text{ terms}} \ .$$

We have the following relationship between $\Phi_g(t)$ and $\mathcal{L}_n(g|t)$.

Proposition 7. *We have, for all non-negative functions g and all $t \in \mathbb{R}^\ell$,*

$$\mathcal{L}_n(g|t) = \exp\left(\left[\Phi_g^n(g)\right](t)\right) \ .$$

The same formula holds if g is complex valued, provided that $\mathcal{L}_n(|g|\,|\,t) < \infty$.

Proof. See Section Appendix A. □

We now consider a function g depending on a second variable $z \in U$. We thus extend the definition of the operator Φ_g to functions h defined on $\mathbb{R}^\ell \times U$ as

$$[\Phi_g(h)](t,z) = g(t,z) + \int (\mathrm{e}^{h(s,z)} - 1)\ p(s-t;s)\mathrm{d}s \quad t \in \mathbb{R},\ z \in U \ , \tag{34}$$

with some adequate conditions on $p(\cdot;\cdot)$, g and h to guarantee that the integral is well defined. In particular, in order to obtain a control of the derivatives of $\mathcal{L}(g(\cdot,z)|t)$ with respect to z, we work within the space $\bar{\mathcal{O}}\,(U)$ by adding some control on adequate norms of the functions (see Section 2.1 where the main notation is introduced). Proposition 7 and (32) immediately provide a way to express $\mathcal{L}(g|t)$.

Corollary 8. *Let $g \in \bar{\mathcal{O}}(U)$. Suppose that there exists a compact set $K \subset U$ and $r_\infty > 0$ such that the sequence $\left(\Phi_g^n(|g|)\right)_{n\geq 1}$ takes its values in $B_{\bar{\mathcal{O}}}(r_\infty; K, \infty)$. Then, we have, for (Lebesgue) almost every $t \in \mathbb{R}^\ell$ and all $z \in K$,*

$$\mathcal{L}(g(\cdot, z)|t) = \lim_{n\to\infty} \exp\left([\Phi_g^n(g)](t,z)\right) \ .$$

The following lemma will be useful.

Lemma 9. *Let $p \in [1, \infty]$. Suppose that $h, h' \in \bar{\mathcal{O}}_p(U) \cap \bar{\mathcal{O}}_\infty(U)$. Then $\mathrm{e}^h - \mathrm{e}^{h'}$ also belong to $\bar{\mathcal{O}}_p(U)$ and, for any compact set K, if $|h|_{\bar{\mathcal{O}},K,\infty} \vee |h'|_{\bar{\mathcal{O}},K,\infty} \leq r_\infty$, we have*

$$\left|\mathrm{e}^h - \mathrm{e}^{h'}\right|_{\bar{\mathcal{O}},K,p} \leq \mathrm{e}^{r_\infty} \ \left|h - h'\right|_{\bar{\mathcal{O}},K,p} \ . \tag{35}$$

Let now $\beta > 0$ and suppose that $h, h' \in \bar{\mathcal{O}}_\infty(U)$. Then, for any compact set K, if $|h|_{\bar{\mathcal{O}},K,\infty} \vee |h'|_{\bar{\mathcal{O}},K,\infty} \leq r_\infty$ and $|h - h'|_{\bar{\mathcal{O}},K,(\beta)} < \infty$, we have

$$\left|\mathrm{e}^h - \mathrm{e}^{h'}\right|_{\bar{\mathcal{O}},K,(\beta)} \leq \mathrm{e}^{r_\infty} \ \left|h - h'\right|_{\bar{\mathcal{O}},K,(\beta)} \ . \tag{36}$$

Proof. This follows directly from the inequality $|\mathrm{e}^x - \mathrm{e}^y| \leq \mathrm{e}^y(y - x)$ valid for all $y \geq x$. ☐

Mimicking the notation introduced in (LS-1) and (LS-4), we consider the following assumption.

(NS-1) We have $\zeta_1 < 1$ and $\zeta_\infty < \infty$ where $\zeta_q = \sup_{t\in\mathbb{R}^\ell} |p(\cdot;t)|_q$.

Recall that the first condition in (NS-1) already appeared in (10) of Definition 1. By Lemma 9, we have that if $h \in \bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$ then $\mathrm{e}^h - 1 \in \bar{\mathcal{O}}_1(U)$. Consequently, if $\zeta_\infty < \infty$, then we get that, for all $t \in \mathbb{R}^\ell$ and compact sets $K \subset U$,

$$\int \sup_{z\in K} \left|\mathrm{e}^{h(s,z)} - 1\right| \ p(s-t;s)\mathrm{d}s \leq \zeta_\infty \left|\mathrm{e}^h - 1\right|_{\bar{\mathcal{O}},K,1} < \infty \ ,$$

and, applying Lemma 15 for any t with μ defined as the measure having density $s \mapsto p(s-t;s)$, it follows that, if $g \in \bar{\mathcal{O}}(U)$, then $\Phi_g(h) \in \bar{\mathcal{O}}(U)$. Applying this line of reasoning, we get the following result.

Proposition 10. *Suppose that (NS-1) holds. Let $g \in \bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$. Then, for all $h \in \bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$, the function $(t,z) \mapsto [\Phi_g(h)](t,z)$ in (34) is well defined on $\mathbb{R}^\ell \times U$ and belong to $\bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$. Moreover, for all $h, h' \in \bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$ and compact sets $K \subset U$,*

(a) $|\Phi_g(h)|_{\bar{\mathcal{O}},K,\infty} \leq |g|_{\bar{\mathcal{O}},K,\infty} + \zeta_\infty \ \left|\mathrm{e}^h - 1\right|_{\bar{\mathcal{O}},K,1}$,

(b) $|\Phi_g(h)|_{\bar{\mathcal{O}},K,1} \leq |g|_{\bar{\mathcal{O}},K,1} + \zeta_1 \ \left|\mathrm{e}^h - 1\right|_{\bar{\mathcal{O}},K,1}$,

(c) $|\Phi_g(h) - \Phi_g(h')|_{\bar{\mathcal{O}},K,1} \leq \zeta_1 \ \left|\mathrm{e}^h - \mathrm{e}^{h'}\right|_{\bar{\mathcal{O}},K,1}$,

We now derive a stability and contraction property on the operator Φ_g for the norms $|\cdot|_{\bar{\mathcal{O}},K,1}$ and $|\cdot|_{\bar{\mathcal{O}},K,\infty}$.

Proposition 11. *Suppose that (NS-1) holds. Let*

$$r_\infty \in (0, -\log \zeta_1) \quad \text{and} \quad r_1 \in \left(0, r_\infty \mathrm{e}^{-r_\infty} \zeta_\infty^{-1}\right) . \tag{37}$$

Then we have

$$R_1 := r_1 \left(1 - \zeta_1 \mathrm{e}^{r_\infty}\right) \in (0, r_1) , \tag{38}$$

$$R_\infty := r_\infty - \mathrm{e}^{r_\infty} \zeta_\infty r_1 \in (0, r_\infty) . \tag{39}$$

Let $K \subset U$ be a compact set and $g \in B_{\bar{\mathcal{O}}}(R_1; K, 1) \cap B_{\bar{\mathcal{O}}}(R_\infty; K, \infty)$. Then $B_{\bar{\mathcal{O}}}(r_1; K, 1) \cap B_{\bar{\mathcal{O}}}(r_\infty; K, \infty)$ is stable for the operator Φ_g, which is strictly contracting on this set for the norm $|\cdot|_{\bar{\mathcal{O}},K,1}$. More precisely, we have

$$\sup \frac{|\Phi_g(h) - \Phi_g(h')|_{\bar{\mathcal{O}},K,1}}{|h - h'|_{\bar{\mathcal{O}},K,1}} \leq \zeta_1 \mathrm{e}^{r_\infty} < 1 ,$$

where the sup is taken over all h, h' in $B_{\bar{\mathcal{O}}}(r_1; K, 1) \cap B_{\bar{\mathcal{O}}}(r_\infty; K, \infty)$ such that $|h - h'|_{\bar{\mathcal{O}},K,1} > 0$.

Proof. Recall that (NS-1) implies $\zeta_1 < 1$. Obviously, (37) then implies $0 < \zeta_1 \mathrm{e}^{r_\infty} < 1$ and then (38) and (39). Let now $K \subset U$ be a compact set, $g \in B_{\bar{\mathcal{O}}}(R_1; K, 1) \cap B_{\bar{\mathcal{O}}}(R_\infty; K, \infty)$ and $h \in B_{\bar{\mathcal{O}}}(r_1; K, 1) \cap B_{\bar{\mathcal{O}}}(r_\infty; K, \infty)$. By Proposition 10 with Lemma 9, we get that

$$\begin{aligned} |\Phi_g(h)|_{\bar{\mathcal{O}},K,1} &\leq |g|_{\bar{\mathcal{O}},K,1} + \zeta_1 \, \mathrm{e}^{r_\infty} \; |h|_{\bar{\mathcal{O}},K,1} \\ &\leq R_1 + \zeta_1 \, \mathrm{e}^{r_\infty} \, r_1 = r_1 . \end{aligned}$$

And, similarly,

$$\begin{aligned} |\Phi_g(h)|_{\bar{\mathcal{O}},K,\infty} &\leq |g|_{\bar{\mathcal{O}},K,\infty} + \zeta_\infty \, \mathrm{e}^{r_\infty} \; |h|_{\bar{\mathcal{O}},K,1} \\ &\leq R_\infty + \zeta_\infty \, \mathrm{e}^{r_\infty} \, r_1 = r_\infty . \end{aligned}$$

Then, $\Phi_g(h) \in B_{\bar{\mathcal{O}}}(r_1; K, 1) \cap B_{\bar{\mathcal{O}}}(r_\infty; K, \infty)$. Finally, using again Proposition 10 with Lemma 9, for all h, h' in $B_{\bar{\mathcal{O}}}(r_1; K, 1) \cap B_{\bar{\mathcal{O}}}(r_\infty; K, \infty)$,

$$\left|\Phi_g(h) - \Phi_g(h')\right|_{\bar{\mathcal{O}},K,1} \leq \zeta_1 \, \mathrm{e}^{r_\infty} \left|h - h'\right|_{\bar{\mathcal{O}},K,1} ,$$

which concludes the proof. $\square$

The stability obtained in Proposition 11 allows one to apply Corollary 8 which says that $\mathcal{L}(g(\cdot, z)|t)$ can be expressed as the limit of $\exp\left([\Phi_g^n(g)](t, z)\right)$ as $n \to \infty$ for all z and almost every t. On the other hand, the space $\bar{\mathcal{O}}_1(U)$ endowed with the convergence in the norm $|\cdot|_{\bar{\mathcal{O}},K,1}$ for all compact sets $K \subset U$ can be made complete by taking equivalent classes for the equivalence relationship $h\mathcal{R}h'$ if $h(t, z) = h'(t, z)$ for all $z \in U$ and almost every $t \in \mathbb{R}^\ell$. Then the standard fixed point theorem shows that $(\Phi_g^n(g))_{n\geq 1}$ converges in $\bar{\mathcal{O}}_1(U)$ to the unique fixed point of Φ_g, and applying Lemma 9, $(\exp(\Phi_g^n(g)))_{n\geq 1}$ in fact converges to $(t, z) \mapsto \mathcal{L}(g(\cdot, z)|t)$ in $\bar{\mathcal{O}}_1(U)$. This is summarized in the following corollary.

Corollary 12. *Suppose that (NS-1) holds. Let $g \in \bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$. The following assertions hold.*

(i) *Let $K \subset U$ be a compact set. If there exist r_1 and r_∞ satisfying (37) such that $g \in B_{\bar{\mathcal{O}}}(R_1; K, 1) \cap B_{\bar{\mathcal{O}}}(R_\infty; K, \infty)$, with R_1, R_∞ defined by (38) and (39), then the sequence $(\Phi_g^n(g))_{n\geq 1}$ takes its values in $B_{\bar{\mathcal{O}}}(r_1; K, 1) \cap B_{\bar{\mathcal{O}}}(r_\infty; K, \infty)$.*

(ii) *Suppose that for all compact sets $K \subset U$, there exist r_1 and r_∞ satisfying (37) such that $g \in B_{\bar{\mathcal{O}}}(R_1; K, 1) \cap B_{\bar{\mathcal{O}}}(R_\infty; K, \infty)$, with R_1, R_∞ defined by (38) and (39). Then $\log \mathcal{L}(g|\cdot)$ satisfies that, for almost every $t \in \mathbb{R}^\ell$, $z \mapsto \mathcal{L}(g(\cdot, z)|t)$ is holomorphic on U and, for all compact sets $K \subset U$,*

$$\lim_{n\to\infty} \int \sup_{z\in K} \left|\mathcal{L}(g(\cdot, z)|t) - \exp\left([\Phi_g^n(g)](t, z)\right)\right| \, \mathrm{d}t = 0 \, .$$

Note in particular that applying (31), Lemma 9 and Lemma 15, this corollary implies that if λ_c is uniformly bounded on $\mathbb{R}^\ell$, then $z \mapsto \mathcal{L}(g(\cdot, z))$ is holomorphic on U.

6.2. *Locally stationary approximation for component point processes*

We now consider a locally stationary Hawkes process $(N_T)_{T>0}$ with local immigrant intensity $\lambda_c^{<\mathrm{LS}>}$ and local fertility function $p^{<\mathrm{LS}>}(\cdot;\cdot)$, see Definition 2. Note that, for any $T > 0$, Assumptions (LS-1) and (LS-4) imply (NS-1) for $p(s;t) = p^{<\mathrm{LS}>}(s; t/T)$. Hence we can apply the results derived in Section 6.1 to the non-stationary Hawkes processes N_T. Also, for any fixed $u \in \mathbb{R}^\ell$, the same assumptions imply (NS-1) for $p(s;t) = p^{<\mathrm{LS}>}(s;u)$ (this $p(s;t)$ does not depend on t) and hence we can also apply the results derived in Section 6.1 to the stationary Hawkes processes $N(\cdot; u)$.

Let us denote by $N_T(\cdot|t)$ and $N(\cdot|t;u)$ the component processes at point t of N_T and $N(\cdot;u)$ and let $\mathcal{L}_T(g|t)$ and $\mathcal{L}(g|t;u)$ denote their Laplace functional, defined as in (30). As in Section 3.1, we will in fact take g depending on two variables $(t, z) \in \mathbb{R}^\ell \times U$ and make the convenient abuse of notation to keep denoting $N_T(g|t)$, $N(g|t;u)$, $\mathcal{L}_T(g|t)$ and $\mathcal{L}(g|t;u)$ the corresponding functions defined on U, that is, for instance, $[N_T(g|t)](z) = N_T(g(\cdot, z)|t)$. And so, continuing the same example, $N_T(g|\cdot)$ is a function defined on $\mathbb{R}^\ell \times U$. The goal of this section is to approximate, for any given $u \in \mathbb{R}^\ell$, $\mathcal{L}_T(\bar{S}^{-Tu} g|t)$ with $\mathcal{L}(g|t;u)$ as $T \to \infty$.

In the locally stationary setting, we use the notation $\zeta_q^{<\mathrm{LS}>}$ introduced in (LS-1) with $q = 1$ and (LS-4) with $q = \infty$ so that the conditions on r_1 and r_∞ in (37) read

$$r_\infty \in (0, -\log \zeta_1^{<\mathrm{LS}>}) \quad \text{and} \quad r_1 \in \left(0, r_\infty \mathrm{e}^{-r_\infty} (\zeta_\infty^{<\mathrm{LS}>})^{-1}\right) \, . \tag{40}$$

and the definition R_1 and R_∞ in (38) and (39) are replaced by

$$R_1 := r_1 \left(1 - \zeta_1^{<\mathrm{LS}>} \mathrm{e}^{r_\infty}\right) \in (0, r_1) \, , \tag{41}$$

$$R_\infty := r_\infty - \mathrm{e}^{r_\infty} \zeta_\infty^{<\mathrm{LS}>} r_1 \in (0, r_\infty) \, . \tag{42}$$

Based on these definitions, we say that $g \in \bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$ satisfies Property (P) if the following holds.

(P) For any compact set $K \subset U$, there exist $r_1(K)$ and $r_\infty(K)$ satisfying (40) such that $g \in B_{\bar{\mathcal{O}}}(R_1(K); K, 1) \cap B_{\bar{\mathcal{O}}}(R_\infty(K); K, \infty)$, with $R_1(K)$, $R_\infty(K)$ defined as in (41) and (42).

We have the following result.

Theorem 13. *Suppose that (LS-1), (LS-3) and (LS-4) hold. Let $\beta \in (0,1]$ and $g \in \bar{\mathcal{O}}_1(U) \cap \bar{\mathcal{O}}_\infty(U)$ satisfying Property (P). Then for all $u \in \mathbb{R}^\ell$ and $T > 0$, and for almost every $t \in \mathbb{R}^\ell$, $z \mapsto \mathcal{L}(g(\cdot,z)|t;u)$ and $z \mapsto \mathcal{L}_T(g(\cdot,z)|t)$ are holomorphic on U. Moreover, for all compact sets $K \subset U$,*

$$\int \sup_{z\in K} \left|\mathcal{L}_T(S^{-Tu}g(\cdot,z)|t) - \mathcal{L}(g(\cdot,z)|t;u)\right| \,\mathrm{d}t \leq A(K)\, T^{-\beta} \left(|g|_{\bar{\mathcal{O}},K,(\beta)} + B(K)\right) , \tag{43}$$

where

$$A(K) = \frac{\left|\xi^{(\beta)}\right|_1 \mathrm{e}^{2\, r_\infty(K)}}{\left(1-\zeta_1^{<LS>}\mathrm{e}^{r_\infty(K)}\right)^2} \quad \text{and} \quad B(K) = r_1(K)\mathrm{e}^{r_\infty(K)}\zeta_{(\beta)}^{<LS>} .$$

Moreover, we have

$$\int \sup_{z\in K} |\mathcal{L}(g(\cdot,z)|t;u) - 1|\; |t|^\beta \,\mathrm{d}t \leq \frac{\mathrm{e}^{r_\infty(K)}}{1-\zeta_1^{<LS>}\mathrm{e}^{r_\infty(K)}} \left(|g|_{\bar{\mathcal{O}},K,(\beta)} + B(K)\right) . \tag{44}$$

The proof of Theorem 13 requires some preliminaries. By Remark 3 and since $g \mapsto \mathcal{L}(g|t;u)$ is translation invariant (for all s, $\mathcal{L}(S^s g|t;u) = \mathcal{L}(g|t;u)$), we can take $u = 0$ without meaningful loss of generality. For convenience, we denote by $p^{<\mathrm{S}>}(t)$ the local fertility function $p^{<\mathrm{LS}>}(t;0)$ at $u = 0$.

Following the definition of Φ_g in (34), we set, for any $g \in \bar{\mathcal{O}}(U)$,

$$[\Phi_{T,g}(h)](t,z) = g(t,z) + \int \left(\mathrm{e}^{h(s,z)} - 1\right)\, p^{<\mathrm{LS}>}(s-t;s/T)\,\mathrm{d}s . \tag{45}$$

$$[\Phi_g^{<\mathrm{S}>}(h)](t,z) = g(t,z) + \int \left(\mathrm{e}^{h(s,z)} - 1\right)\, p^{<\mathrm{S}>}(s-t)\,\mathrm{d}s . \tag{46}$$

The following lemma will be useful.

Lemma 14. *Let $\beta \in (0,1]$. Suppose that (LS-1) and (LS-4) hold and define r_1 and r_∞ as in (37). Let $g \in B_{\bar{\mathcal{O}}}(R_1;K,1) \cap B_{\bar{\mathcal{O}}}(R_\infty;K,\infty)$ with R_1 and R_∞ defined by (38) and (39) respectively. Let $r_{(\beta)}$ be a constant satisfying*

$$r_{(\beta)} > (1-\mathrm{e}^{r_\infty}\zeta_1^{<LS>})^{-1} r_1 \mathrm{e}^{r_\infty}\zeta_{(\beta)}^{<LS>} . \tag{47}$$

Then we have

$$R_{(\beta)} := r_{(\beta)}(1-\mathrm{e}^{r_\infty}\zeta_1^{<LS>}) - r_1\mathrm{e}^{r_\infty}\zeta_{(\beta)}^{<LS>} \in (0, r_{(\beta)}) . \tag{48}$$

Moreover, for all compact sets $K \subset U$, if $g \in B_{\bar{\mathcal{O}}}(R_1;K,1) \cap B_{\bar{\mathcal{O}}}(R_\infty;K,\infty) \cap B_{\bar{\mathcal{O}}}\left(R_{(\beta)};K,(\beta)\right)$, then $B_{\bar{\mathcal{O}}}(r_1;K,1) \cap B_{\bar{\mathcal{O}}}(r_\infty;K,\infty) \cap B_{\bar{\mathcal{O}}}\left(r_{(\beta)};K,(\beta)\right)$ is stable for the operator $\Phi_g^{<S>}$.

Proof. Let $K \subset U$ be a compact set. Suppose that $g \in B_{\bar{\mathcal{O}}}(R_1;K,1) \cap B_{\bar{\mathcal{O}}}(R_\infty;K,\infty) \cap B_{\bar{\mathcal{O}}}\left(R_{(\beta)};K,(\beta)\right)$. We already know from Proposition 10 that then $B_{\bar{\mathcal{O}}}(r_1;K,1) \cap B_{\bar{\mathcal{O}}}(r_\infty;K,\infty)$ is stable for the operator Φ_g. Let now $h \in B_{\bar{\mathcal{O}}}(r_1;K,1) \cap B_{\bar{\mathcal{O}}}(r_\infty;K,\infty) \cap B_{\bar{\mathcal{O}}}\left(r_{(\beta)};K,(\beta)\right)$. Then we have

$$\begin{aligned}
\left|\Phi_g^{<\mathrm{S}>}(h)\right|_{\bar{\mathcal{O}},K,(\beta)} &\leq |g|_{\bar{\mathcal{O}},K,(\beta)} + \int \sup_{z\in K} \left|\int \left(\mathrm{e}^{h(s,z)}-1\right) p^{<\mathrm{S}>}(s-t)\,\mathrm{d}s\right|\; |t|^\beta\,\mathrm{d}t \\
&\leq |g|_{\bar{\mathcal{O}},K,(\beta)} + \int \sup_{z\in K} \left|\mathrm{e}^{h(s,z)}-1\right| \left(\int p^{<\mathrm{S}>}(s-t)\; |t|^\beta \mathrm{d}t\right)\,\mathrm{d}s .
\end{aligned}$$

Observe that, using that $|r-s|^\beta \le |r|^\beta + |s|^\beta$ for $\beta \in (0,1]$, we have, for all $s \in \mathbb{R}^\ell$,

$$\int p^{\text{<S>}}(s-t)\,|t|^\beta \mathrm{d}t = \int p^{\text{<S>}}(r)\,|r-s|^\beta \mathrm{d}t \le \left|p^{\text{<S>}}\right|_{(\beta)} + \left|p^{\text{<S>}}\right|_1\,|s|^\beta\,.$$

Inserting this bound in the previous display and using Lemma 9, we get that

$$\begin{aligned}\left|\Phi_g^{\text{<S>}}(h)\right|_{\bar{\mathcal{O}},K,(\beta)} &\le |g|_{\bar{\mathcal{O}},K,(\beta)} + \mathrm{e}^{r_\infty}\left|p^{\text{<S>}}\right|_{(\beta)}|h|_{\bar{\mathcal{O}},K,1} + \mathrm{e}^{r_\infty}\,\left|p^{\text{<S>}}\right|_1\,|h|_{\bar{\mathcal{O}},K,(\beta)} \\ &\le R_{(\beta)} + \mathrm{e}^{r_\infty}\zeta_{(\beta)}^{\text{<LS>}}\,r_1 + \mathrm{e}^{r_\infty}\,\zeta_1^{\text{<LS>}}\,r_{(\beta)} = r_{(\beta)}\,,\end{aligned}$$

where the equality follows from (48). □

We can now prove Theorem 13 in the case $u=0$.

Proof of Theorem 13. We deduce from the preliminaries that Proposition 11 and Corollary 12 apply for each $T>0$ and each $u \in \mathbb{R}^d$. Thus for almost every $t \in \mathbb{R}^\ell$, $z \mapsto \mathcal{L}(g(\cdot,z)|t;u)$ and $z \mapsto \mathcal{L}_T(g(\cdot,z)|t)$ are holomorphic on U and it only remains to prove the bound (43) for a given compact set $K \subset U$, again picking the case $u=0$ without loss of generality, in which case we denote $\mathcal{L}^{\text{<S>}}(g|t) = \mathcal{L}(g|t;0)$. We suppose that

$$|g|_{\bar{\mathcal{O}},K,(\beta)} < \infty\,. \tag{49}$$

(Otherwise the right-hand side of (43) is infinite and there is nothing to prove.) Then by assumption on g and Proposition 11,

$$B := B_{\bar{\mathcal{O}}}\left(r_1(K);K,1\right) \cap B_{\bar{\mathcal{O}}}\left(r_\infty(K);K,\infty\right)$$

is stable both for $\Phi_{T,g}$ and $\Phi_g^{\text{<S>}}$ and moreover these operators are Lipschitz for the $|\cdot|_{\bar{\mathcal{O}},K,1}$-norm with Lipschitz constant

$$\rho := \zeta_1^{\text{<LS>}}\mathrm{e}^{r_\infty} < 1\,.$$

Let us now write, for any $n \ge 1$ and all $h \in B$,

$$\left|\Phi_{T,g}^n(h) - \Phi_g^{\text{<S>}\,n}(h)\right|_{\bar{\mathcal{O}},K,1} \le \sum_{k=0}^{n-1}\left|\Phi_{T,g}^{n-k}\circ\Phi_g^{\text{<S>}\,k}(h) - \Phi_{T,g}^{n-k-1}\circ\Phi_g^{\text{<S>}\,k+1}(h)\right|_{\bar{\mathcal{O}},K,1}\,.$$

Using the Lipschitz property of $\Phi_{T,g}$ in B, we get, for all $h \in B$,

$$\left|\Phi_{T,g}^n(h) - \Phi_g^{\text{<S>}\,n}(h)\right|_{\bar{\mathcal{O}},K,1} \le \sum_{k=0}^{n-1}\rho^{n-k-1}\left|\Phi_{T,g}\circ\Phi_g^{\text{<S>}\,k}(h) - \Phi_g^{\text{<S>}\,k+1}(h)\right|_{\bar{\mathcal{O}},K,1}\,. \tag{50}$$

Using (LS-3), we have, for all $h \in B$,

$$\begin{aligned}\left|\Phi_{T,g}(h) - \Phi_g^{\text{<S>}}(h)\right|_{\bar{\mathcal{O}},K,1} &= \int \sup_{z\in K}\left|\int (\mathrm{e}^{h(s,z)}-1)\left[p^{\text{<LS>}}(s-t;s/T) - p^{\text{<LS>}}(s-t;0)\right]\,\mathrm{d}s\right|\,\mathrm{d}t \\ &\le T^{-\beta}\int \sup_{z\in K}\left|\int (\mathrm{e}^{h(s,z)}-1)\,\xi^{(\beta)}(s-t)|s|^\beta\,\mathrm{d}s\right|\,\mathrm{d}t \\ &\le T^{-\beta}\,\left|\xi^{(\beta)}\right|_1\,\left|\mathrm{e}^h - 1\right|_{\bar{\mathcal{O}},K,(\beta)}\,.\end{aligned}$$

Using Lemma 9 and inserting this in (50), we get, for all $h \in B$,

$$\left|\Phi_{T,g}^{n}(h) - \Phi_{g}^{<\mathrm{S}>\, n}(h)\right|_{\bar{\mathcal{O}},K,1} \leq T^{-\beta}\ \left|\xi^{(\beta)}\right|_{1}\ \mathrm{e}^{r_\infty}\ \sum_{k=0}^{n-1} \rho^{n-k-1} \left|\Phi_{g}^{<\mathrm{S}>\, k}(h)\right|_{\bar{\mathcal{O}},K,(\beta)}\ . \tag{51}$$

By Condition (49) and since $\rho = \zeta_1^{<\mathrm{LS}>} \mathrm{e}^{r_\infty} < 1$, we have

$$\left(1 - \zeta_1^{<\mathrm{LS}>} \mathrm{e}^{r_\infty}\right)^{-1} r_1 \mathrm{e}^{r_\infty} \zeta_{(\beta)}^{<\mathrm{LS}>} \leq \left(1 - \zeta_1^{<\mathrm{LS}>} \mathrm{e}^{r_\infty}\right)^{-1} \left(|g|_{\bar{\mathcal{O}},K,(\beta)} + r_1 \mathrm{e}^{r_\infty} \zeta_{(\beta)}^{<\mathrm{LS}>}\right) < \infty\ ,$$

and thus, for all

$$r_{(\beta)} > \left(1 - \zeta_1^{<\mathrm{LS}>} \mathrm{e}^{r_\infty}\right)^{-1} \left(|g|_{\bar{\mathcal{O}},K,(\beta)} + r_1 \mathrm{e}^{r_\infty} \zeta_{(\beta)}^{<\mathrm{LS}>}\right)\ , \tag{52}$$

the $R_{(\beta)}$ defined by (48) is such that $|g|_{\bar{\mathcal{O}},K,(\beta)} < R_{(\beta)}$. Then Lemma 14 gives that the set $B_{\bar{\mathcal{O}}}\left(r_1; K, 1\right) \cap B_{\bar{\mathcal{O}}}\left(r_\infty; K, \infty\right) \cap B_{\bar{\mathcal{O}}}\left(r_{(\beta)}; K, (\beta)\right)$ is stable for the operator $\Phi_g^{<\mathrm{S}>}$. We thus have, for all $h \in B_{\bar{\mathcal{O}}}\left(r_1; K, 1\right) \cap B_{\bar{\mathcal{O}}}\left(r_\infty; K, \infty\right) \cap B_{\bar{\mathcal{O}}}\left(r_{(\beta)}; K, (\beta)\right)$ and $k \geq 0$,

$$\left|\Phi_{g}^{<\mathrm{S}>\, k}(h)\right|_{\bar{\mathcal{O}},K,(\beta)} \leq r_{(\beta)} \tag{53}$$

We thus get from (51) that, for all $h \in B_{\bar{\mathcal{O}}}\left(r_1; K, 1\right) \cap B_{\bar{\mathcal{O}}}\left(r_\infty; K, \infty\right) \cap B_{\bar{\mathcal{O}}}\left(r_{(\beta)}; K, (\beta)\right)$, we have

$$\left|\Phi_{T,g}^{n}(h) - \Phi_{g}^{<\mathrm{S}>\, n}(h)\right|_{\bar{\mathcal{O}},K,1} \leq T^{-\beta}\ \left|\xi^{(\beta)}\right|_{1}\ \mathrm{e}^{r_\infty}\ r_{(\beta)} (1-\rho)^{-1}\ .$$

To conclude, we apply this to $h = g$ (since by construction $g \in B_{\bar{\mathcal{O}}}\left(R_1; K, 1\right) \cap B_{\bar{\mathcal{O}}}\left(R_\infty; K, \infty\right) \cap B_{\bar{\mathcal{O}}}\left(R_{(\beta)}; K, (\beta)\right) \subset B_{\bar{\mathcal{O}}}\left(r_1; K, 1\right) \cap B_{\bar{\mathcal{O}}}\left(r_\infty; K, \infty\right) \cap B_{\bar{\mathcal{O}}}\left(r_{(\beta)}; K, (\beta)\right)$ and let $r_{(\beta)}$ tend to the right-hand side of (52) and obtain that, for all $n \geq 1$,

$$\left|\Phi_{T,g}^{n}(g) - \Phi_{g}^{<\mathrm{S}>\, n}(g)\right|_{\bar{\mathcal{O}},K,1} \leq T^{-\beta}\ \frac{\left|\xi^{(\beta)}\right|_{1}\ \mathrm{e}^{r_\infty} \left(|g|_{\bar{\mathcal{O}},K,(\beta)} + r_1 \mathrm{e}^{r_\infty} \zeta_{(\beta)}^{<\mathrm{LS}>}\right)}{\left(1 - \zeta_1^{<\mathrm{LS}>} \mathrm{e}^{r_\infty}\right)^2}\ .$$

With Lemma 9, it yields that, for all $n \geq 1$,

$$\left|\exp\left(\Phi_{T,g}^{n}(g)\right) - \exp\left(\Phi_{g}^{<\mathrm{S}>\, n}(g)\right)\right|_{\bar{\mathcal{O}},K,1} \leq T^{-\beta}\ \frac{\left|\xi^{(\beta)}\right|_{1}\ \mathrm{e}^{2\, r_\infty} \left(|g|_{\bar{\mathcal{O}},K,(\beta)} + r_1 \mathrm{e}^{r_\infty} \zeta_{(\beta)}^{<\mathrm{LS}>}\right)}{\left(1 - \zeta_1^{<\mathrm{LS}>} \mathrm{e}^{r_\infty}\right)^2}\ .$$

Applying Corollary 12, we thus obtain (43) for all compact sets $K \subset U$.

The bound (44) is a by product of the above proof. Namely, observe that by Corollary 8 and Fatou's lemma, we have

$$\begin{aligned}\int \sup_{z \in K} \left|\mathcal{L}^{<\mathrm{S}>}\left(g(\cdot, z) | t\right) - 1\right|\ |t|^\beta\ \mathrm{d}t &= \int \sup_{z \in K} \lim_{n \to \infty} \left|\exp\left([\Phi_g^{<\mathrm{S}>\, n}(g)](t, z)\right) - 1\right|\ |t|^\beta\ \mathrm{d}t \\ &\leq \liminf_{n \to \infty} \left|\exp \Phi_g^{<\mathrm{S}>\, n}(g) - 1\right|_{\bar{\mathcal{O}},K,(\beta)}\ .\end{aligned}$$

Now recall that we already used that $B_{\bar{\mathcal{O}}}\left(r_1; K, 1\right) \cap B_{\bar{\mathcal{O}}}\left(r_\infty; K, \infty\right) \cap B_{\bar{\mathcal{O}}}\left(r_{(\beta)}; K, (\beta)\right)$ is stable for the operator $\Phi_g^{<\mathrm{S}>}$, so with Lemma 9 and the previous bound we get

$$\int \sup_{z \in K} \left|\mathcal{L}^{<\mathrm{S}>}\left(g(\cdot, z) | t\right) - 1\right|\ |t|^\beta\ \mathrm{d}t \leq \mathrm{e}^{r_\infty}\ r_{(\beta)}\ .$$

Letting $r_{(\beta)}$ tend to the right-hand side of (52) as above we get (44) in the case $u = 0$, which concludes the proof. $\square$

6.3. Local Laplace functional

We use the same notation as in Sections 6.1 and 6.2. Let us first explain how to use the previous results (mainly Proposition 11 and Theorem 13) for deriving the Laplace functional $\mathcal{L}_T(S^{-Tu}g)$ of N_T and the Laplace functional $\mathcal{L}(\cdot;u)$ of the stationary Hawkes process $N(\cdot;u)$. We again set $u=0$ in the following without loss of meaningful generality and denote $\mathcal{L}^{\text{<S>}} = \mathcal{L}(\cdot;0)$, $\mathcal{L}^{\text{<S>}}(g|\cdot) = \mathcal{L}(g|\cdot;0)$, $\lambda_c^{\text{<S>}} = \lambda_c^{\text{<LS>}}(0)$ and $p^{\text{<S>}} = p^{\text{<LS>}}(\cdot;0)$.

We suppose that the assumptions of Theorem 13 hold for some given function g. Let $\Phi_{T,g}^{\infty}$ and $\Phi_g^{\text{<S>}\,\infty}$ denote the two fixed point limits given by Proposition 11, which, for all compact sets $K\subset U$, are elements of the stable set $B_{\bar{\mathcal{O}}}(r_1(K);K,1)\cap B_{\bar{\mathcal{O}}}(r_\infty(K);K,\infty)$. Note that Proposition 10 shows that $\exp\left(\Phi_{T,g}^{\infty}\right)-1$ is essentially bounded on $\mathbb{R}^\ell\times K$ for all compact set $K\subset U$. Hence, from Corollary 8 and applying (31), we get that if $|\lambda_c^{\text{<LS>}}|_\infty<\infty$, for all $T>0$,

$$\mathcal{L}_T(g) = \exp\int\left(\exp\left(\Phi_{T,g}^{\infty}(t,\cdot)\right)-1\right)\,\lambda_c^{\text{<LS>}}(t/T)\,\mathrm{d}t\ ,$$

and

$$\mathcal{L}^{\text{<S>}}(g) = \exp\left(\lambda_c^{\text{<S>}}\int\left(\exp\left(\Phi_g^{\text{<S>}\,\infty}(t,\cdot)\right)-1\right)\,\mathrm{d}t\right)\ ,$$

and by Lemma 15, these two functions are holomorphic on U. We thus define $\mathcal{K}_T(g)$ and $\mathcal{K}(g;u)$ by

$$\mathcal{K}_T(g) = \int\left(\exp\left(\Phi_{T,g}^{\infty}(t,\cdot)\right)-1\right)\,\lambda_c^{\text{<LS>}}(t/T)\,\mathrm{d}t$$

and

$$\mathcal{K}^{\text{<S>}}(g) = \mathcal{K}(g;0) = \lambda_c^{\text{<S>}}\int\left(\exp\left(\Phi_g^{\text{<S>}\,\infty}(t,\cdot)\right)-1\right)\,\mathrm{d}t$$

Now we observe that, for any compact set $K\subset U$,

$$\begin{aligned}\left|\mathcal{K}_T(g)-\mathcal{K}^{\text{<S>}}(g)\right|_{\mathcal{O},K} &\leq \sup_{z\in K}\left|\int\left(\exp\left(\Phi_{T,g}^{\infty}(t,z)\right)-\exp\left(\Phi_g^{\text{<S>}\,\infty}(t,z)\right)\right)\,\lambda_c^{\text{<LS>}}(t/T)\,\mathrm{d}t\right| \\ &\quad+\sup_{z\in K}\left|\int\left(\exp\left(\Phi_g^{\text{<S>}\,\infty}(t,z)\right)-1\right)\,\left(\lambda_c^{\text{<LS>}}(t/T)-\lambda_c(0)\right)\mathrm{d}t\right| =: \text{(I)}+\text{(II)}\ .\end{aligned}$$

We can bound (I) as

$$\begin{aligned}\text{(I)} &\leq \left|\lambda_c^{\text{<LS>}}\right|_\infty\,\left|\exp\left(\Phi_{T,g}^{\infty}\right)-\exp\left(\Phi_g^{\text{<S>}\,\infty}\right)\right|_{\bar{\mathcal{O}},K,1} \\ &= \left|\lambda_c^{\text{<LS>}}\right|_\infty\int\sup_{z\in K}\left|\mathcal{L}_T(g(\cdot,z)|t)-\mathcal{L}^{\text{<S>}}(g(\cdot,z)|t)\right|\,\mathrm{d}t\ .\end{aligned}$$

Using (LS-2), the term (II) is easily bounded as

$$\begin{aligned}\text{(II)} &\leq \xi_c^{(\beta)}\,T^{-\beta}\sup_{z\in K}\int\left|\exp\left(\Phi_g^{\text{<S>}\,\infty}(t,z)\right)-1\right|\,|t|^\beta\mathrm{d}t \\ &= \xi_c^{(\beta)}\,T^{-\beta}\sup_{z\in K}\int\left|\mathcal{L}^{\text{<S>}}(g(\cdot,z)|t)-1\right|\,|t|^\beta\mathrm{d}t \\ &\leq \xi_c^{(\beta)}\,T^{-\beta}\int\sup_{z\in K}\left|\mathcal{L}^{\text{<S>}}(g(\cdot,z)|t)-1\right|\,|t|^\beta\mathrm{d}t\ .\end{aligned}$$

We can now bound (I) and (II) by relying on Theorem 13, so that

$$(\mathrm{I})+(\mathrm{II}) \leq T^{-\beta}\left\{\left|\lambda_c^{<\mathrm{LS}>}\right|_\infty A(K)+\xi_c^{(\beta)} \frac{\mathrm{e}^{r_\infty(K)}}{1-\zeta_1^{<\mathrm{LS}>} \mathrm{e}^{r_\infty(K)}}\right\}\left(|g|_{\bar{\mathcal{O}}, K,(\beta)}+B(K)\right), \tag{54}$$

provided that the assumptions of Theorem 13 hold. Hence, the proof of Theorem 2 now boils down to the following.

Proof of Theorem 2. As explained above, we just need to prove that the assumptions of Theorem 13 holds. The only non-trivial one is to prove that g satisfies Property (P). Let $K \subset U$ be compact. We set

$$r_\infty(K)=-\frac{1}{2} \log \zeta_1^{<\mathrm{LS}>},$$

which by (LS-1) satisfies the left-hand side condition of (40). Then the right-hand side condition on $r_1(K)$ reads

$$0<r_1(K)<r_\infty(K)(\zeta_1^{<\mathrm{LS}>})^{1/2}(\zeta_\infty^{<\mathrm{LS}>})^{-1}, \tag{55}$$

and $R_1(K)$ and $R_\infty(K)$ defined by (41) and (42) are given by

$$R_1(K)=r_1(K)\left(1-(\zeta_1^{<\mathrm{LS}>})^{1/2}\right) \quad \text{and} \quad R_\infty(K)=r_\infty(K)-(\zeta_1^{<\mathrm{LS}>})^{-1/2} \zeta_\infty^{<\mathrm{LS}>} r_1(K).$$

Condition (17) and the choice of $r_\infty(K)$ above implies

$$a:=\frac{|g|_{\bar{\mathcal{O}}, K, 1}}{\left(1-(\zeta_1^{<\mathrm{LS}>})^{1/2}\right)}<r_\infty(K)(\zeta_1^{<\mathrm{LS}>})^{1/2}(\zeta_\infty^{<\mathrm{LS}>})^{-1}=: b,$$

Now, any $r_1(K)$ strictly being between these two boundaries satisfies (55) and the corresponding $R_1(K)$ satisfies $|g|_{\bar{\mathcal{O}}, K, 1}<R_1(K)$. Moreover as $r_1(K)$ tends to the lower boundary a from above, we have

$$R_\infty(K) \uparrow r_\infty(K)-(\zeta_1^{<\mathrm{LS}>})^{-1/2} \zeta_\infty^{<\mathrm{LS}>} \frac{|g|_{\bar{\mathcal{O}}, K, 1}}{1-(\zeta_1^{<\mathrm{LS}>})^{1/2}}.$$

From (18), we obtain that $|g|_{\bar{\mathcal{O}}, K, \infty}<R_\infty(K)$ for $r_1(K)$ chosen close enough to a. Hence we have shown that g satisfies Property (P) and the proof is concluded. The constants C_1 and C_2 in (20) correspond to the $\{\dots\}$ term in (54) and $B(K)$ with the above definitions of $r_\infty(K)$ and $r_1(K)$. ☐

6.4. Local cumulants

Proof of Theorem 4. We apply Theorem 2 first with $g(t, z)=z\, h(t)$, defined on $(t, z) \in \mathbb{R}^{\ell} \times \mathbb{C}$ and then with

$$g(t, z)=\sum_{j=1}^m z_j\, g_j(t) \tag{56}$$

defined on $(t, z) \in \mathbb{R}^{\ell} \times \mathbb{C}^m$. The fact that $N_T(h)$ and $N(h ; u)$ admit finite exponential moments for a bounded integrable function $g: \mathbb{R}^{\ell} \rightarrow \mathbb{R}$ is a direct application of Theorem 2 for the first choice of g.

We now apply the theorem with g defined as in (56). We assume such that $|g_j|_{(\beta)} < \infty$ for all $j = 1, \ldots, m$ (otherwise the right-hand side of the inequality is infinite and there is nothing to prove). Take U the polydisc $P_r^m(0)$ of $\mathbb{C}^m$ with center 0 and radius $r > 0$. In this case we have, for any compact set $K \subset U$ and any $q \in [1, \infty]$,

$$|g|_{\bar{\mathcal{O}},K,q} < r \sum_{j=1}^{m} |g_j|_q \ .$$

Hence (17) and (18) hold for r small enough so that the two following inequalities hold.

$$r \sum_{j=1}^{m} |g_j|_1 \leq \left(-\frac{1}{2} \log \zeta_1^{<\mathrm{LS}>}\right) (\zeta_1^{<\mathrm{LS}>})^{1/2} (\zeta_\infty^{<\mathrm{LS}>})^{-1} (1 - \zeta_1^{<\mathrm{LS}>})^{1/2} \ ,$$
$$r \sum_{j=1}^{m} |g_j|_\infty \leq -\frac{1}{2} \log \zeta_1^{<\mathrm{LS}>} - (\zeta_1^{<\mathrm{LS}>})^{-1/2} (\zeta_\infty^{<\mathrm{LS}>}) (1 - \zeta_1^{<\mathrm{LS}>})^{-1/2} r \sum_{j=1}^{m} |g_j|_1 \ .$$

The largest r satisfying these two conditions is easily found to be

$$r := \frac{(-\log \zeta_1^{<\mathrm{LS}>}/2)}{\sum_{j=1}^m |g_j|_\infty + (\zeta_1^{<\mathrm{LS}>})^{-1/2} \zeta_\infty^{<\mathrm{LS}>} (1 - \zeta_1^{<\mathrm{LS}>})^{-1/2} \sum_{j=1}^m |g_j|_1} \ .$$

Moreover we also have

$$|g|_{\bar{\mathcal{O}},K,(\beta)} < r \sum_{j=1}^{m} |g_j|_{(\beta)} \ .$$

Hence Theorem 2 with (14), the above bounds on $|g|_{\bar{\mathcal{O}},K,1}$ and $|g|_{\bar{\mathcal{O}},K,(\beta)}$, and the Cauchy inequality (2), imply

$$\begin{aligned} &\left|\mathrm{Cum}\left(N_T(S^{-Tu} g_1), \ldots, N_T(S^{-Tu} g_m)\right) - \mathrm{Cum}\left(N(g_1; u), \ldots, N(g_m; u)\right)\right| \\ &\qquad \leq r_0^{1-m}\, C_1 \sum_{j=1,\ldots,m} \left(|g_j|_{(\beta)} + C_2 \, |g_j|_1\right) \, T^{-\beta} \ , \end{aligned}$$

for any $r_0 \in (0, r)$. Letting r_0 tend to r, this bound is still valid with r_0^{1-m} replaced by

$$\left(\frac{\sum_{j=1}^m |g_j|_\infty + (\zeta_1^{<\mathrm{LS}>})^{-1/2} \zeta_\infty^{<\mathrm{LS}>} (1 - \zeta_1^{<\mathrm{LS}>})^{-1/2} \sum_{j=1}^m |g_j|_1}{(-\log \zeta_1^{<\mathrm{LS}>}/2)}\right)^{m-1} \ .$$

This concludes the proof. □

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Appendix A. A postponed proof and a useful lemma

Proof of Proposition 7. By denoting by $\mathcal{F}_j$ the σ-algebra generated by the family $(N^{(k)})_{0\le k\le j}$, we have

$$\mathcal{L}_n(g|t) = \mathbb{E}\left[\exp\sum_{k=0}^{n} N^{(k)}(g|t)\right] = \mathbb{E}\left[\exp\sum_{k=0}^{n-1} N^{(k)}(g|t) + \mathbb{E}\left[\exp N^{(n)}(g|t)\,|\,\mathcal{F}_{n-1}\right]\right].$$

Since conditionally on $\mathcal{F}_{n-1}$, $N^{(n)}(\cdot|t)$ is a sum of independent PPP's with intensities $s\mapsto p(s-r;s)$ with r describing all points of $N^{(n-1)}(\cdot|t)$, we have for any $h:\mathbb{R}^\ell\to\mathbb{R}_+$,

$$\mathbb{E}\left[\exp N^{(n)}(h|t)\,|\,\mathcal{F}_{n-1}\right] = \exp\left(\int (\mathrm{e}^{h(s)}-1)p(s-r;s)\,\mathrm{d}s\, N^{(n-1)}(\mathrm{d}r|t)\right).$$

Applying this with the definition of Φ_g and iterating, we get

$$\begin{aligned}
&\mathbb{E}\left[\exp\sum_{k=0}^{n-1} N^{(k)}(g|t) + \mathbb{E}\left[\exp N^{(n)}(h|t)\,|\,\mathcal{F}_{n-1}\right]\right]\\
&= \mathbb{E}\left[\exp\left(\sum_{k=0}^{n-2} N^{(k)}(g|t) + N^{(n-1)}([\Phi_g(h)]\,|\,t)\right)\right]\\
&= \mathbb{E}\left[\exp\left(\sum_{k=0}^{n-3} N^{(k)}(g|t) + N^{(n-2)}([\Phi_g\circ\Phi_g(h)]\,|\,t)\right)\right]\\
&\vdots\\
&= \mathbb{E}\left[\exp\left(N^{(0)}([\Phi_g^n(h)]\,|\,t)\right)\right]\\
&= \exp\left([\Phi_g^n(h)](t)\right)\ .
\end{aligned}$$

Applying the obtained formula with $h=g$, we obtain the claimed result. $\square$

The following lemma is a straightforward application of the Cauchy inequality (2).

Lemma 15. *Let μ be a non-negative measure on $\mathbb{R}^\ell$ and $h\in\bar{\mathcal{O}}(U)$. Suppose that for all $z\in U$, there exists a neighborhood $V\subset U$ of z such that*

$$\mu\left(\sup_{z\in V} h(\cdot,z)\right) < \infty\ .$$

Then $z\mapsto\mu(h(\cdot,z))$ belongs to $\mathcal{O}(U)$ and for any multi-index α, we have, for all $z\in U$,

$$\partial^\alpha\mu(h(\cdot,z)) = \mu\left(\partial^\alpha_{\mathcal{O}} h(\cdot,z)\right)\ .$$

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