Computation of the probability of error for HARQ-CC with macro-diversity based on MRC.

Tania Alhajj, Xavier Lagrange

To cite this version:

Tania Alhajj, Xavier Lagrange. Computation of the probability of error for HARQ-CC with macro-diversity based on MRC.. [Research Report] RR-2020-01-SRCD, IMT ATLANTIQUE. 2020. hal-02949519

HAL Id: hal-02949519
https://hal-imt.archives-ouvertes.fr/hal-02949519
Submitted on 25 Sep 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Computation of the probability of error for HARQ-CC with macro-diversity based on MRC.

Tania Alhajj (IMT Atlantique)
Xavier Lagrange (IMT Atlantique)
Computation of the probability of error for HARQ-CC with macro-diversity based on MRC.

Tania Alhajj
IMT Atlantique
IRISA, UMR CNRS 6074
F-35700 Rennes, France
tania.alhajj@imt-atlantique.fr

Xavier Lagrange
IMT Atlantique
IRISA, UMR CNRS 6074
F-35700 Rennes, France
xavier.lagrange@imt-atlantique.fr

Abstract

In this report, we study a protocol with hybrid automatic repeat request (HARQ) when it is combined with macro-diversity. We derive the analytic formula of the probability of error using HARQ-Chase Combining (HARQ-CC) with the maximum ratio combining (MRC) technique for the macro-diversity.

I. INTRODUCTION

Using HARQ for error correction has been widely known in the mobile networks. HARQ increases the reliability of the communication. The HARQ process has two types: HARQ-Chase Combining (HARQ-CC) and HARQ-Incremental Redundancy (HARQ-IR). In HARQ-CC, a packet is sent to the receiver. The receiver tries to decode it. If the decoding attempt succeeds, an acknowledgement (ACK) message is sent to the sender. Otherwise, a negative acknowledgement (NACK) is sent to the sender asking for a re-transmission. The re-transmission consists on re-transmitting exactly the same packet. The receiver combines the two versions of the packet: the old/erroneous one and the new one and tries to decode again. This process is repeated until a maximum number of transmissions is reached or until the decoding success. Previous studies were made to evaluate the error while using HARQ-CC. In [1], the author evaluated the probability of error as a function of the average signal to noise ratio (SNR) using HARQ-CC over a fast-fading Rayleigh channel.

The aim of diversity is to combat the channels with bad conditions. Macro-diversity is a type of spatial diversity where there are multiple transmitters and/or multiple receivers. In our case, we consider the case of multiple receivers. With macro-diversity, the receivers are separated and present on different cell sites. Multiple receptions thus have to be combined in a centralized entity using a combining diversity technique. One of the combining techniques is maximum ratio combining (MRC). This technique consists on summing the received signals together before the decoding process.

In the present work, we combine the HARQ with macro-diversity. We derive the analytic formula of the probability of error using HARQ-CC with the MRC combining technique for the macro-diversity. We provide the detailed calculations steps.

II. MODEL OVERVIEW

We consider transmissions in the uplink (UL) direction. We consider that one user equipment (UE) is transmitting and two base station (BS)s are receiving. The channel suffers fast fading. Thus, the SNR is an exponential random variable with mean the average SNR: $\gamma$. In [2], the packet error rate (PER) as a function of the SNR was provided:

$$h(\gamma) = \begin{cases} 
1 & \text{if } 0 < \gamma < \gamma_M \\
\alpha e^{-\gamma g} & \text{if } \gamma \geq \gamma_M 
\end{cases}$$

(1)

where $\alpha$ and $g$ are parameters that are modulation and coding scheme (MCS) mode dependent and $\gamma_M = \ln(\alpha)/g$.

III. ANALYTIC DERIVATION

For the MRC, during each transmission, the received SNR is the sum of the two SNRs received at each BS as shown in section 13.4-1 in [3]:

$$\gamma_{S,i} = \gamma_{1,i} + \gamma_{2,i}$$

(2)

with $\gamma_{m,i}$ being the SNR received from the UE at BS$_m$ during the $i$th transmission.

If $\gamma_1 << \gamma_2$, then $\gamma_{S,i} \approx \gamma_{2,i}$. Similarly, if $\gamma_2 << \gamma_1$, then $\gamma_{S,i} \approx \gamma_{1,i}$. We thus take the case where $\gamma_1 = \gamma_2 = \gamma$. We have $\gamma_{m,i}$ an exponential random variable: $\gamma_{m,i} \sim \exp(\frac{1}{\gamma_m})$. Then, $\gamma_{S,i} = \gamma_{1,i} + \gamma_{2,i}$ follows the Erlang law with the following distribution:

$$f_\gamma(\gamma_{S,i}) = \left(\frac{1}{\gamma}\right)^2 \gamma_{S,i} e^{-\frac{1}{\gamma}\gamma_{S,i}}$$

(3)
Using HARQ-CC, the probability of error during the $k$th transmission depends on the SNR of all the $k$ transmissions: it is $h(\gamma_{T,k})$ where $\gamma_{T,k} = \sum_{i=1}^{k} \gamma_{S,i}$.

We consider successive packet transmissions, so the SNR is independent and identically distributed (i.i.d) for different transmissions. Therefore, the probability of error during the $k$th transmission has to be averaged over all $\gamma_{S,i}$ with $i : 1 \rightarrow k$.

Using HARQ-CC, the probability of error during the $k$th transmission is based on all the SNRs perceived by the receiver from the first transmission until the $k$th one. Consequently, the probability of needing more than $k$ transmissions has to be averaged over all $\gamma_{S,i}$.

Finally, the probability of having an error at the $k$th transmission is:

$$P(l > k) = \prod_{i=1}^{k} h(\gamma_{S,i}) f_{\gamma_{S,1}}(\gamma_{S,1}) ... f_{\gamma_{S,k}}(\gamma_{S,k}) d\gamma_{S,1} ... d\gamma_{S,k}.$$  

Similarly to [1], we split (4) into the sum of three series:

$$P(l > k) = A_k + \sum_{l=1}^{k-1} B_{k,l} + C_k$$  

where:

$$A_k = \int_{\gamma}^{2\gamma} \int_{\gamma}^{2\gamma} \sum_{i=1}^{k} f_{\gamma_{S,1}}(\gamma_{S,1}) ... f_{\gamma_{S,k}}(\gamma_{S,k}) d\gamma_{S,1} ... d\gamma_{S,k}$$

$$B_{k,l} = \int_{\gamma}^{2\gamma} \int_{\gamma}^{2\gamma} \sum_{j=l+1}^{k} f_{\gamma_{S,1}}(\gamma_{S,1}) ... f_{\gamma_{S,k}}(\gamma_{S,k}) a e^{-\gamma_{T,l}} ... a e^{-\gamma_{T,k}} d\gamma_{S,1} ... d\gamma_{S,k}$$

$$C_k = \int_{\gamma}^{2\gamma} \int_{\gamma}^{2\gamma} \sum_{j=1}^{l} f_{\gamma_{S,1}}(\gamma_{S,1}) ... f_{\gamma_{S,k}}(\gamma_{S,k}) a e^{-\gamma_{T,l}} ... a e^{-\gamma_{T,k}} d\gamma_{S,1} ... d\gamma_{S,k}.$$  

The calculation steps in Section IV lead to the following expressions:

$$A_k = 1 - e^{-\frac{2\gamma}{\gamma M}} \sum_{n=0}^{2k-1} \left( \frac{\gamma M}{\gamma} \right)^n \frac{1}{n!}$$

$$B_{k,l} = e^{-\frac{2\gamma}{\gamma M}} \frac{1}{(2l+1)!} \left( \frac{\gamma M}{\gamma} \right)^{2l} \prod_{j=l+1}^{k} \left( 1 + g(k + j) \right)$$

$$C_k = e^{-\frac{2\gamma}{\gamma M}} \prod_{j=1}^{l} \left( 1 + g(k + j) \right)$$

Finally, the probability of having an error at the $k$th transmission while using HARQ-CC and MRC based macro-diversity is:

$$P(l > k) = 1 - e^{-Y} \sum_{i=0}^{k-1} \frac{1}{i!} + \sum_{l=0}^{k-1} e^{-Y} Y^{2l} \prod_{j=1}^{k-l} \frac{1}{(1+\gamma g_j)^2} E_{k,l}$$  

where $Y = \frac{2\gamma}{\gamma M}$ and $E_{k,l} = \gamma M \left( 1 + g(k + l) \right) + 2l + 1$.

### IV. Detailed Derivation of $A_k$, $B_{k,l}$, and $C_k$

#### A. Computation of $A_k$

$$A_k = \int_{\gamma}^{2\gamma} \int_{\gamma}^{2\gamma} ... \int_{\gamma}^{2\gamma} f_{\gamma_{S,1}}(\gamma_{S,1}) ... f_{\gamma_{S,k}}(\gamma_{S,k}) d\gamma_{S,1} ... d\gamma_{S,k}.$$  

Now, we make a variable change by dividing $\gamma_{S,i}$ by $\gamma_{M}$:

$$A_k = \left( \frac{\gamma M}{\gamma} \right)^{2k} \int_{0}^{1} \int_{0}^{1} ... \int_{0}^{1} \gamma_{S,1} \gamma_{S,2} ... \gamma_{S,k} \left( e^{-\frac{\gamma_{S,1}}{\gamma M}} e^{-\frac{\gamma_{S,2}}{\gamma M}} ... e^{-\frac{\gamma_{S,k}}{\gamma M}} \right) d\gamma_{S,1} d\gamma_{S,2} ... d\gamma_{S,k}.$$  

Using the multiple integral expression from [4, eq. 2 pp. 614], we get:

$$A_k = \frac{1}{\Gamma(2k)} \left( \frac{\gamma M}{\gamma} \right)^{2k} \int_{0}^{1} e^{-x^{2k-1}} dx.$$  

The previous integral can be solved using [4, 11 §2.33, pp. 108]. We finally get:

$$A_k = 1 - e^{-\frac{2\gamma}{\gamma M}} \sum_{n=0}^{2k-1} \left( \frac{\gamma M}{\gamma} \right)^n \frac{1}{n!}.$$  


B. Computation of $B_{k,l}$

$$B_{k,l} = \int_0^{\gamma M} \cdots \int_0^{\gamma M - \gamma_{T,l-1}} \int_0^{\gamma M - \gamma_{T,l}} \cdots \int_0^{\gamma M} f_\gamma(\gamma S_1) \cdots f_\gamma(\gamma S_k) ae^{-g \gamma_{T,l+1}} \cdots ae^{-g \gamma_{T}} e^{d \gamma S_1 \cdots d \gamma S_k}$$

where $\gamma_{T,k} = \sum_{i=1}^k \gamma S_i$.

For $i > l + 1$:

$$\int_0^{\gamma M} \cdots \int_0^{\gamma M - \gamma_{T,l}} f_\gamma(\gamma S_{l+2}) \cdots f_\gamma(\gamma S_k) ae^{-g \gamma_{T,l+2}} \cdots ae^{-g \gamma_{T}} e^{d \gamma S_{l+2} \cdots d \gamma S_k} = \frac{\alpha}{(1 + g \gamma(k + 1 - i))^2}.$$ 

So we have:

$$\int_0^{\gamma M} \cdots \int_0^{\gamma M - \gamma_{T,l}} \gamma S_{l+2} \cdots \gamma S_k ae^{-g \gamma_{T,l+2}} \cdots ae^{-g \gamma_{T}} e^{d \gamma S_{l+2} \cdots d \gamma S_k} \prod_{i=l+2}^k \frac{\alpha}{(1 + g \gamma(k + 1 - i))^2}.$$ 

Then, for $i > l$:

$$U_{k,l} = \int_0^{\gamma M - \gamma_{T,l}} f_\gamma(\gamma S_{l+1}) ae^{-g(k-l) \gamma S_{l+1}} \prod_{i=l+2}^k \frac{\alpha}{(1 + g \gamma(k + 1 - i))^2} d \gamma S_{l+1}.$$ 

After several elementary computation steps we get:

$$U_{k,l} = V_{k,l} e^{g \gamma_{T,l}(g(k-l)+\frac{\gamma}{7})} \left[ 1 + \left( \frac{\gamma}{7} + g(k-l) \right) (\gamma M - \gamma_{T,l}) \right]$$

with $V_{k,l} = \prod_{i=l+1}^k \left( \frac{\gamma}{7 + g \gamma(k + 1 - i)} \right)^{e^{-2\gamma M}}$.

So,

$$B_{k,l} = \frac{1}{\gamma M} V_{k,l} \int_0^{\gamma M} \cdots \int_0^{\gamma M - \gamma_{T,l-1}} \gamma S_{l+1} \cdots \gamma S_k e^{d \gamma S_{l+1} \cdots d \gamma S_k}$$

$$+ \frac{1}{\gamma M} \left( \frac{\gamma}{7} + g(k-l) \right) V_{k,l} \int_0^{\gamma M} \cdots \int_0^{\gamma M - \gamma_{T,l-1}} \gamma S_{l+1} \cdots \gamma S_k e^{d \gamma S_{l+1} \cdots d \gamma S_k}$$

where

$$I_l = \frac{1}{\gamma M} \int_0^{\gamma M} \cdots \int_0^{\gamma M - \gamma_{T,l-1}} \gamma S_{l+1} \cdots \gamma S_k e^{d \gamma S_{l+1} \cdots d \gamma S_k} = \frac{\gamma M^{2l}}{(2l)!} \quad (11)$$

and

$$J_l = \frac{\gamma M}{7} \frac{\gamma M^{2l+1}}{(2l+1)(2l-1)!} \quad (12)$$

Finally,

$$B_{k,l} = e^{-\gamma M} \frac{1}{(2l+1)!} \left( \frac{\gamma M}{7} \right)^{2l} \prod_{j=l+1}^k \frac{1}{(1 + g \gamma(k + 1 - j))^2} \left[ \gamma M \left( \frac{\gamma}{7} + g(k-l) \right) + 2l + 1 \right] \quad (13)$$

1) Proof of (11): We prove (11) by induction. By definition of $I_l$, $I_1$ is:

$$I_1 = \int_0^{\gamma M} \gamma S_1 e^{d \gamma S_1} = \frac{\gamma M^2}{2}.$$ 

Thus, (11) is valid for $l = 1$. We suppose that:

$$I_{l-1} = \frac{\gamma M^{2l-1}}{(2l-1)!}$$

and we denote it as $I_{l-1}(\gamma M)$ for the sake of clarity in the next steps. Now, we check if it remains true for $l$:

$$I_l = \int_0^{\gamma M} \gamma S_1 \int_0^{\gamma M - \gamma S_1} \cdots \int_0^{\gamma M - \gamma_{T,l-1}} \gamma S_k e^{d \gamma S_{l+1} \cdots d \gamma S_k}$$

$$I_l = \int_0^{\gamma M} \gamma S_1 I_{l-1}(\gamma M - \gamma S_1) e^{d \gamma S_1}.$$
I_l = \int_0^{\gamma_M} \gamma_{S,l} \frac{(\gamma_M - \gamma_{S,l})^{2l-2}}{(2l-2)!} d\gamma_{S,l} \\quad I_l = \frac{\gamma_M^{2l}}{(2l)!} \\

2) Proof of (12):

J_l = \int_0^{\gamma_M} \int_0^{\gamma_M} \cdots \int_0^{\gamma_M} \gamma_{S,1} \gamma_{S,2} \cdots \gamma_{S,l} (-\gamma_{S,1} - \gamma_{S,2} - \cdots - \gamma_{S,l}) d\gamma_{S,1} d\gamma_{S,2} \cdots d\gamma_{S,l}.

Like for (11), we make the proof by induction for (12). By definition of J_l, J_1 is:

J_1 = \int_0^{\gamma_M} \gamma_{S,1} (-\gamma_{S,1}) d\gamma_{S,1} = -\frac{\gamma_M^3}{3}.

So, (12) is valid for l = 1. We suppose that

J_{l-1} = -\frac{\gamma_M^{2l-1}}{(2l-1)(2l-3)!}

and we denote it as J_{l-1}(\gamma_M). Now, we check if it is still true for l:

\begin{align*}
J_l &= \int_0^{\gamma_M} -\gamma_{S,1}^2 \int_0^{\gamma_M} \cdots \int_0^{\gamma_M} \gamma_{S,2} \cdots \gamma_{S,l} d\gamma_{S,1} d\gamma_{S,2} \cdots d\gamma_{S,l} \\
&\quad + \int_0^{\gamma_M} \gamma_{S,1} \int_0^{\gamma_M} \cdots \int_0^{\gamma_M} \gamma_{S,2} \cdots \gamma_{S,l} (-\gamma_{S,2} - \cdots - \gamma_{S,l}) d\gamma_{S,1} d\gamma_{S,2} \cdots d\gamma_{S,l} \\
J_l &= \int_0^{\gamma_M} -\gamma_{S,1}^2 J_{l-1}(\gamma_M - \gamma_{S,1}) d\gamma_{S,1} + \int_0^{\gamma_M} \gamma_{S,1} J_{l-1}(\gamma_M - \gamma_{S,1}) d\gamma_{S,1} \\
J_l &= \int_0^{\gamma_M} -\gamma_{S,1}^2 \frac{(\gamma_M - \gamma_{S,1})^{2l-1}}{(2l-1)!} d\gamma_{S,1} + \int_0^{\gamma_M} \gamma_{S,1} \left( \frac{(\gamma_M - \gamma_{S,1})^{2l-1}}{(2l-1)(2l-3)!} \right) d\gamma_{S,1} \\
J_l &= -\frac{\gamma_M^{2l+1}}{(2l+1)(2l-1)!}.
\end{align*}

C. Computation of C_k

\begin{align*}
C_k &= B_{k,0} = U_{k,0} \\
C_k &= \prod_{j=2}^{k} \frac{1}{(1 + g^{\gamma}(k+1-j))^2} \int_{\gamma_M}^\infty \frac{\gamma_{S,1}}{\gamma^{2}} e^{-\frac{\gamma}{\gamma}} e^{-g(k-l)\gamma_{S,1}} d\gamma_{S,1} \\
C_k &= e^{-2\gamma} \prod_{j=1}^{k} \frac{1}{(1 + g^{\gamma}(k+1-j))^2} \left[ 1 + \gamma M \left( \frac{1}{g^\gamma} + g_{k} \right) \right].
\end{align*}

V. Conclusion

In this work, we provided the detailed analytic formulation for the probability of error while using HARQ-CC with an order 2 macro-diversity combined by the MRC technique.

REFERENCES


Campus de Brest
Technopôle Brest-Iroise
CS 83818
29238 Brest Cedex 3
France
T +33 (0)2 29 00 11 11
F +33 (0)2 29 00 10 00

Campus de Nantes
4, rue Alfred Kastler
CS 20722
44307 Nantes Cedex 3
France
T +33 (0)2 51 85 81 00
F +33 (0)2 99 12 70 08

Campus de Rennes
2, rue de la Châtaigneraie
CS 17607
35576 Cesson Sévigné Cedex
France
T +33 (0)2 99 12 70 00
F +33 (0)2 51 85 81 99

Site de Toulouse
10, avenue Édouard Belin
BP 44004
31028 Toulouse Cedex 04
France
T +33 (0)5 61 33 83 65

IMT Atlantique
Bretagne-Pays de la Loire
École Mines-Télécom

IMT Atlantique, 2020